THE ODD IMAGE THEOREM

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ABSTRACT

Let M be an oriented space-time, let T be a time-like path, and let x be a point in M. We define an integer $\deg(T, x)$ which we call the *platonic degree* of the history T and the observer x. This platonic degree is closely related to the number of future pointing null geodesics from T to x. In the case of a smooth T the number of images of a source on T has the same parity as the platonic degree for almost all x. This degree can be calculated by a winding number argument which is very intuitive.

1. Degree of Maps

Let $f : M \to N$ be a continuous map between oriented n dimensional manifolds. The local degree of f at $y \in N$ denoted $\deg(f)_y$ is defined by the homomorphism $f_* : H_n(M, M - f^{-1}(y)) \to H_n(N, N - y)$. Now $\deg(f)_y$ depends only upon a neighborhood of y and the homotopy class of f. If M and N are compact with no boundaries, the degree of f , denoted $\deg(f)$, is defined by $f_* : H_n(M) \to H_n(N)$. Then $\deg(f)_y = \deg(f)$ for all y in N. If the boundaries of M and N are not empty and if f restricts to a map $g : \partial M \to \partial N$ then $\deg(f)_y = \deg(g)$ for all y in N. If M is compact with boundary and N is homeomorphic to n dimensional Euclidean space $\mathbf{R^n}$, and if y is not in the image of ∂M under f, we can define a map $h : \partial M \to S^{n-1}$ by $h(x) = (f(x) - y)/||f(x) - y||$. The *winding number of* ∂M *about* y is defined to be the degree of h. This winding number depends on y. The winding number about y equals the local degree at y of f.

2. Platonic Degree

Let $\tau \in T$. The set of future pointing null geodesics originating at τ are in one-one correspondence to the points of a two-sphere S^2. Let S be a space-like slice containing x such that every future null geodesic originating on T crosses S exactly once. Define the continuous map $f : S^2 \to S$ by sending the point on S^2 to the point on S where the corresponding geodesic intersects S. Then putting together all the f for each $\tau \in T$, we get a continuous map $F : S^2 \times T \to S$. We call this a *platonic map* because it resonates on several levels with Plato's famous discription of reality as shadows projected on the wall of a cave.

Assume that no future null geodesic hits x from the endpoints of T. The local degree of F at x is the integer generating the image of

$$F_* : H_3(S^2 \times T, S^2 \times T - F^{-1}(x)) \to H_3(S, S - x)$$

Local degrees can be read about in Albrecht Dold, Lectures on Algebraic Topology, Springer-Verlag, New York, (1972), see Chapter IV, Section 5.

Definition: We define $\deg(T, x)$ by setting it equal to the local degree of F at x.

This is independent of all the choices made. First we must specify how we combine the celestial spheres along T to make $S^2 \times T$. For example we could choose to parallel translate a S^2 along T, or we could Fermi-Walker translate S^2 along T, etc. The different choices can be represented by a homeomorphism from $S^2 \times T$ to $S^2 \times T$. This will have degree one and will not affect the local degree since the degrees of compositions of maps multiply. Second, we can choose a different slice S' through x which gives us a different map F'. Since we can deform S' onto S, at least locally, the homotopy invariance of degree gives us the same local degree. We also note that we can define $\deg(T, x)$ in the case where x is contained in an open neighborhood N so that every future pointing null geodesic from T which intersects N does not stop in N and also has an open neighborhood of geodesics which intersect N. So S need not be global and M need not be complete in order to define the platonic degree.

3. Platonic waves

Now we suppose that T and S are smooth, so F is smooth. Local degree and winding numbers are easy to compute in the smooth setting. Let τ_0 be the beginning of T and let τ_1 be the end of T. Let the *platonic wave front at* τ be the image of F restricted to the τ slice in $S^2 \times T$. As τ runs from τ_1 to τ_0 this wave front expands through S. If τ_1 is in S, the waves will begin at a point source.

Let s be a point in $S^2 \times T$. We choose a tangent frame P, E, B, N at s. Here P is the future time-like vector pointing along the T direction and N is space-like outward normal to the celestial sphere at s. We pick E and B tangent to the sphere so that the frame's orientation agrees with that of M. Now F carries the first three vectors to P', E', B', which are tangent to S at $F(s)$. Now P', E', B' do not necessarily form a frame at $F(s)$, but at all but a set of measure zero the P', E', B' do form a frame. This is a consequence of Sard's Lemma.

Now we imagine this platonic wave evolving in S. The vector P' is the velocity vector of the wave front. Note that P' need not be orthogonal to the wave front. We assume that x is a regular point of F as this will be true for all but a set of measure zero in S. Thus the platonic wave fronts pass over x with P', E', B' a frame agreeing or disagreeing with the orientation of S induced from that of M. The local degree of F at x is equal to the number of times the wave front passed over x with positive orientation minus the number of times the wave front passed over x with negative orientation. Thus $\deg(T, x)$ has the same parity as the number of null geodesics from T to x.

Now if S is diffeomorphic to Euclidean space $\mathbf{R^3}$, or even if the platonic wave only passes over a neigborhood diffeomorphic to $\mathbf{R^3}$, we can use winding numbers to calculate $\deg(T, x)$. Send a line out to infinity from x which passes through the beginning and the ending wave fronts at points where E', B' are linealy independent, so that the velocity vector V of the line makes a three frame V, E', B'. Such points are dense. Then the number of points so that V, E', B' have positive orientation minus the number of points where V, E', B' have negative orientation gives the winding number of the boundary of $S^2 \times T$ about x. Hence this is equal to $\deg(T, x)$. If the end of T is in S and is not x, then need only consider the winding number of the τ_0 wave front. If there is a line out to infinity which intersects that wave front once in a general position, then the number of images seen by x of the source T is odd.

Note that in a static space-time, the platonic wave can be taken as the projection of the "real wave". The more the space-time is time dependent, the less the "real wave" makes sense. But the platonic wave always makes mathematical sense even though it is motion in a platonic time occurring in a slice of frozen time.

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