THE LEADING IDEAL OF A COMPLETE INTERSECTION OF HEIGHT TWO IN A 2-DIMENSIONAL REGULAR LOCAL RING

SHIRO GOTO, WILLIAM HEINZER, AND MEE-KYOUNG KIM

ABSTRACT. Let $(S, \mathbf{n})$ be a 2-dimensional regular local ring and let $I = (f, g)$ be an ideal in S generated by a regular sequence f, g of length two. Let I^* be the leading ideal of I in the associated graded ring $\mathrm{gr}_{\mathfrak{n}}(S)$, and set $R = S/I$ and $\mathfrak{m} = \mathfrak{n}/I$. In [GHK2], we prove that if $\mu_G(I^*) = n$, then I^* contains a homogeneous system $\{\xi_i\}_{1 \leq i \leq n}$ of generators such that $\deg \xi_i + 2 \leq \deg \xi_{i+1}$ for $2 \leq i \leq n-1$, and $\mathrm{ht}_G(\xi_1, \xi_2, \cdots, \xi_{n-1}) = 1$, and we describe precisely the Hilbert series $\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda)$ in terms of the degrees c_i of the ξ_i and the integers d_i, where d_i is the degree of $D_i = \mathrm{GCD}(\xi_1, \ldots, \xi_i)$. To the complete intersection ideal $I = (f, g)S$ we associate a positive integer n with $2 \leq n \leq c_1 + 1$, an ascending sequence of positive integers $(c_1, c_2, \ldots, c_n)$, and a descending sequence of integers $(d_1 = c_1, d_2, \ldots, d_n = 0)$ such that $c_{i+1} - c_i > d_{i-1} - d_i > 0$ for each i with $2 \leq i \leq i - 1$. We establish here that this necessary condition is also sufficient for there to exist a complete intersection ideal $I = (f, g)$ whose leading ideal has these invariants. We give several examples to illustrate our theorems.

1. INTRODUCTION

This paper examines generators of the leading ideal of a complete intersection of height two in a 2-dimensional regular local ring. Motivation for our work comes from a paper of S. C. Kothari [K] that answers several questions raised by Abhyankar concerning the local Hilbert function of a pair of plane curves. Before going ahead, let us fix some notation, which we shall maintain throughout this paper.

Setting 1.1. Let $(S, \mathfrak{n})$ be a regular local ring of dimension 2 and let $I = (f, g)$ be an ideal in S generated by a regular sequence f, g of length two. For simplicity we assume that the residue class field $k = S/\mathfrak{n}$ is infinite. We put $R = S/I$ and $\mathfrak{m} = \mathfrak{n}/I$. Let

$$\mathrm{R}'(\mathfrak{n}) = \sum_{i \in \mathbb{Z}} \mathfrak{n}^i t^i \subseteq S[t, t^{-1}] \quad \text{and} \quad \mathrm{R}'(\mathfrak{m}) = \sum_{i \in \mathbb{Z}} \mathfrak{m}^i t^i \subseteq R[t, t^{-1}]$$

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denote the Rees algebras of $\mathfrak{n}$ and $\mathfrak{m}$ respectively, where t is an indeterminate. We put

$$G = \mathrm{gr}_{\mathfrak{n}}(S) = \mathrm{R}'(\mathfrak{n})/t^{-1}\mathrm{R}'(\mathfrak{n}) \quad \text{and} \quad \mathrm{gr}_{\mathfrak{m}}(R) = \mathrm{R}'(\mathfrak{m})/t^{-1}\mathrm{R}'(\mathfrak{m}).$$

For each $0 \neq h \in S$ let $\mathrm{o}(h) = \sup\{i \in \mathbb{Z} \mid h \in \mathfrak{n}^i\}$ and put $h^* = \overline{ht^n}$, where $n = \mathrm{o}(h)$ and $\overline{ht^n}$ denotes the image of ht^n in G. The canonical map $S \to R$ induces the epimorphism $\varphi : G \to \mathrm{gr}_{\mathfrak{m}}(R)$ of the associated graded rings. We put

$$I^* = \mathrm{Ker}\ (G \xrightarrow{\varphi} \mathrm{gr}_{\mathfrak{m}}(R)).$$

Then the homogeneous components $\{[I^*]_i\}_{i \in \mathbb{Z}}$ of the leading form ideal I^* of I are given by

$$[I^*]_i = \{\overline{ht^i} \mid h \in I \cap \mathfrak{n}^i\}$$

for each $i \in \mathbb{Z}$. We throughout assume that $a = \mathrm{o}(f) \leq b = \mathrm{o}(g)$ and that $f^* \nmid g^*$ in G. The latter part of the condition is equivalent to saying that f^*, g^* form a part of a minimal homogeneous system of generators of I^*.

Let $\ell_S(*)$ denote length over S. Kothari in [K] proves that

$$0 \leq \dim_k[\mathrm{gr}_{\mathfrak{m}}(R)]_i - \dim_k[\mathrm{gr}_{\mathfrak{m}}(R)]_{i+1} \leq 1$$

for all $i \geq a$ and that $\ell_S(R) \geq ab$; moreover, one has the equality $\ell_S(R) = ab$ if and only if $f^*,\ g^*$ are coprime in G, that is, $f^*,\ g^*$ form a G-regular sequence.

F. Macaulay in a 1904 paper [M] employs a different method to determine the same necessary condition as Kothari on the Hilbert function of a pair of plane curves. Using his inverse systems, Macaulay establishes the structure of the Hilbert function $H(A)$ of a complete intersection quotient $A = k[[x, y]]/(f, g)$ to be of the form

$$H = (1, 2, \ldots, a, t_a, \ldots, t_j, 0), \tag{1}$$

where $a \geq t_a \geq t_{a+1} \geq \cdots \geq t_j = 1$ and $|t_i - t_{i+1}| \leq 1$ for all i. Thus the Hilbert function H after an initial rising segment breaks up into platforms and regular flights of descending stairs, each step of height one. The structure of $H(A)$ is studied from the point of view of parametrizations by J. Briançon [Br] and by A. Iarrobino [Ia1] and [Ia2]. These authors prove that every sequence satisfying the conditions in Equation 1 is realizable as the Hilbert function $H(A)$ of some Gorenstein Artin algebra of the form $A = k[[x, y]]/(f, g)$.

Let $v(H) = 2 + \#\{\text{platforms}\}$. Iarrobino [Ia1], [Ia2] proves that I^* needs two initial generators f^*, g^* and requires a new generator following each platform, and that $v(H)$ is the minimum possible number of generators of a graded ideal defining a standard algebra with Hilbert function H. In [Ia1, Theorem 2.2.A], Iarrobino characterizes those graded ideals corresponding to I^* for which I is a complete intersection of height two. He proves they are exactly the graded ideals with $v(H)$ generators. In the paper [GHK2] we prove the following.

Theorem 1.2. [GHK2, Theorem 1.2] *Let notation be as in Setting 1.1 and assume that $n = \mu_G(I^*)$. Then I^* contains a homogeneous system $\{\xi_i\}_{1\le i\le n}$ of generators that satisfy the following three conditions.*

(1) $\xi_1 = f^*$ *and* $\xi_2 = g^*$.
(2) $\deg\xi_i + 2 \le \deg\xi_{i+1}$ *for each i with* $2 \le i \le n-1$.
(3) $\mathrm{ht}_G(\xi_1, \xi_2, \cdots, \xi_{n-1}) = 1$.

Let $\{\xi_i\}_{1\le i\le n}$ be a homogeneous system of generators of I^* satisfying conditions (1) and (2) in Theorem 1.2. Then the ideals $\{(\xi_j \mid 1 \le j \le i)G\}_{1\le i\le n}$ are independent of the particular choice of the family $\{\xi_i\}_{1\le i\le n}$ and are uniquely determined by I. We put $D_i = \mathrm{GCD}(\xi_j \mid 1 \le j \le i)$ and $d_i = \deg D_i$. We then have the descending sequence of integers

$$a = d_1 > d_2 > \cdots > d_{n-1} > d_n = 0,$$

and we also have $\dfrac{\xi_{i+1}}{D_{i+1}} \in (\dfrac{\xi_1}{D_i}, \dfrac{\xi_2}{D_i}, \cdots, \dfrac{\xi_i}{D_i})$ for all $1 \le i \le n-1$ [GHK2, Lemma 3.2]. Let $c_i = \deg\, \xi_i$ and let $\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda) = \sum_{i=0}^{\infty} \dim_k [\mathrm{gr}_{\mathfrak{m}}(R)]_i \lambda^i$ denote the Hilbert series of $\mathrm{gr}_{\mathfrak{m}}(R)$. Theorem 1.3 explicitly describes $\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda)$ and the difference $\ell_S(R) - ab$ in terms of the integers c_i and d_i, thus sharpening Kothari's results.

Theorem 1.3. [GHK2, Theorem 1.3] *Let notation be as in Setting 1.1 and assume that $n = \mu_G(I^*)$. The following assertions hold true.*

(1) $\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda) = \frac{\sum_{i=2}^{n} \lambda^{d_i}(1-\lambda^{d_{i-1}-d_i})(1-\lambda^{c_i-d_i})}{(1-\lambda)^2}$.
(2) $\ell_S(R) = \sum_{i=2}^{n}(d_{i-1} - d_i)(c_i - d_i) = ab + \sum_{i=2}^{n-1} d_i{\cdot}[(c_{i+1} - c_i) - (d_{i-1} - d_i)]$.
(3) $c_{i+1} - c_i > d_{i-1} - d_i > 0$ *for each i with* $2 \le i \le n-1$.
(4) [K, Corollary1] $\ell_S(R) = ab$ *if and only if $n = 2$, i.e., f^*, g^* is a G-regular sequence.*

In Section 2, we use the inverse of a transformation considered by Kothari and described in parts (1) and (2) of [GHK2, Corollary 2.5] to establish the existence of examples showing that every Hilbert series described in Theorem 1.3 is realizable as the Hilbert series of $\mathrm{gr}_{\mathfrak{m}}(R)$ for some complete intersection ideal $I = (f, g)$. Thus the conditions given in Theorem 1.3 are both necessary and sufficient for there to exist a complete intersection ideal $I = (f, g)$ with Hilbert series

$$\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda) = \frac{\sum_{i=2}^{n} \lambda^{d_i}(1 - \lambda^{d_{i-1}-d_i})(1 - \lambda^{c_i - d_i})}{(1-\lambda)^2}.$$

Remark 1.4. In the case where f and g are a regular sequence in a regular local ring S with $\dim S > 2$, it is still true that $\mathrm{ht}_G(f^*, g^*) > 1$ implies f^*, g^* is a G-regular sequence, and therefore $I^* = (f^*, g^*)G$ also in this case. Thus if $\mathrm{ht}_G(f^*, g^*) = 1$ and if we set $D_2 = \mathrm{GCD}(f^*, g^*)$ and $d_2 = \deg D_2$, then $f^* = D_2\xi$ and $g^* = D_2\eta$. Notice that ξ, η is a regular sequence in G. We have $b \geq a > d_2 > 0$, and $\mu_G(I^*) = n \geq 3$. There exists a minimal homogeneous system $\{\xi_1, \xi_2, \dots, \xi_n\}$ of generators of I^* such that $\xi_1 = f^*$ and $\xi_2 = g^*$, and $c_i := \deg \xi_i \leq \deg \xi_{i+1} := c_{i+1}$ for each $i \leq n-1$. However, the ideal I^* may fail to be perfect, and it is possible to have $D_3 := \mathrm{GCD}(\xi_1, \xi_2, \xi_3) = D_2$ as is illustrated in [GHK1, Example 1.6]. We prove in [GHK1, Theorem 1.2] that I^* is perfect if $n = 3$. We also prove in [GHK1] that $\xi_3 = h^*$, where h has the form $h = \alpha f + \beta g \in I$ with $\mathrm{o}(\alpha) = b - d_2$, and $\mathrm{o}(\beta) = a - d_2$, and that $c_3 := \mathrm{o}(h) > a + b - d_2$. Moreover, if $q = \sigma f + \tau g$ is such that $q^* \notin (f^*, g^*)G$ and $(o)(\sigma) = b - d_2$, then $\mathrm{o}(q) = \mathrm{o}(h)$ and $(f^*, g^*, h^*)G = (f^*, g^*, q^*)G$. Thus the ideal $(\xi_1, \xi_2, \xi_3)G$ is independent of the choice of ξ_3. In the case where $n \geq 4$, we also prove that $c_4 \geq c_3 + 2$ [GHK1, Proposition 2.4]. However, examples shown to us by Craig Huneke and Lance Bryant show that it is possible to have $c_{i+1} = c_i$ for $i \geq 4$. This resolves a question mentioned in [GHK1, Discussion 2.5]. If I^* is perfect, we prove in [GHK2] that $c_{i+1} \geq c_i + 2$ for each i with $2 \leq i \leq n-1$. A question raised in [GHK2] that remains open is whether for I^* perfect in this higher dimensional setting, does it follow that $I^* \subseteq (\xi_1/D_2, \xi_2/D_2)G$.

2. The Main Results

We record in Proposition 2.1 behavior of the Hilbert function with respect to an inverse of a transformation considered by Kothari and described in parts (1) and (2) of [GHK2, Corollary 2.5].

Proposition 2.1. *Assume notation as in Setting 1.1 and let $n = \mu_G(I^*)$. Let $\xi_1, \xi_2, \cdots, \xi_n$ be a minimal homogeneous system of generators for I^* satisfying conditions (1) and (2) in Theorem 1.2, and let $c_i = \deg \xi_i$ for $1 \leq i \leq n$. Also assume that $\mathfrak{n} = (x, y)$ and $x^* \nmid f^*$. Let m be a positive integer and consider the ideal $J_m := (f, x^m g)S$. Then the following assertions hold true.*

(1) $\mu_G(J_m^*) = \mu_G(I^*)$.

(2) J_m^* *has* $\xi_1, X^m\xi_2, X^m\xi_3, \dots, X^m\xi_n$ *as a minimal homogeneous system of generators, so the degree sequence for* J_m^* *is* $(c_1', c_2', \dots, c_n')$*, where* $c_1' = c_1$ *and* $c_i' = c_i + m$ *for* $i \geq 2$.

(3) *With* $D_i' = \mathrm{GCD}(\xi_1, X^m\xi_2, \dots, X^m\xi_i)$ *and* $d_i' = \deg D_i'$*, we have* $D_i' = D_i$ *and* $d_i' = d_i$ *for* $1 \leq i \leq n$.

(4) $\mathrm{H}(\mathrm{gr}_{\mathfrak{n}}(S/J_m), \lambda) = \dfrac{\sum_{i=2}^n \lambda^{d_i}(1-\lambda^{d_{i-1}-d_i})(1-\lambda^m\lambda^{c_i-d_i})}{(1-\lambda)^2}$.

(5) $\ell(S/J_m) - \ell(S/I) = ma$.

Since the residue field of S is infinite, with notation as in Setting 1.1 we may choose x, y so that $\mathfrak{n} = (x, y)$ and $x^* \nmid f^*$ and $y^* \nmid f^*$. Thus it is possible to obtain the hypothesis of Theorem 2.2.

Theorem 2.2. *Assume notation as in Setting 1.1 with* $n = \mu_G(I^*)$*,* $a < b$ *and* $\mathfrak{n} = (x, y)$ *such that* $x^* \nmid f^*$ *and* $y^* \nmid f^*$*. Let* $\xi_1, \xi_2, \cdots, \xi_n$ *be a minimal homogeneous system of generators for* I^* *satisfying conditions (1) and (2) in Theorem 1.2 and let* $c_i = \deg \xi_i$ *for* $1 \leq i \leq n$*. Let* m *be a positive integer and set*

$$V_m = (y^m f,\ x^m(f+g))S.$$

Then the following assertions hold true.

(1) V_m *is* $\mathfrak{n}$*-primary and* $\mu_G(V_m^*) = \mu_G(I^*) + 1$.

(2) V_m^* *has* $Y^m\xi_1, X^m\xi_1, X^mY^m\xi_2, X^mY^m\xi_3, \dots, X^mY^m\xi_n$ *as a minimal homogeneous system of generators, so the degree sequence for* V_m^* *is* $(c_1', c_2', \dots, c_{n+1}')$*, where* $c_1' = c_1 + m$*,* $c_2' = c_1 + m$ *and* $c_i' = c_{i-1} + 2m$ *for* $i \geq 3$.

(3) *With* $D_i' = \mathrm{GCD}(Y^m\xi_1, X^m\xi_1, X^mY^m\xi_2, \dots, X^mY^m\xi_{i-1})$ *and* $d_i' = \deg D_i'$*, we have* $D_1' = Y^m D_1$*, and* $D_i' = D_{i-1}$ *for* $2 \leq i \leq n+1$*. Thus* $d_1' = m + d_1$ *and* $d_i' = d_{i-1}$ *for* $2 \leq i \leq n+1$.

(4)
$$\begin{aligned}\mathrm{H}(\mathrm{gr}_{\mathfrak{n}}(S/V_m), \lambda) &= \frac{\sum_{i=2}^{n+1} \lambda^{d_i'}(1-\lambda^{d_{i-1}'-d_i'})(1-\lambda^{c_i'-d_i'})}{(1-\lambda)^2} \\ &= \lambda^a(1+\lambda+\lambda^2+\cdots+\lambda^{m-1})^2 + \frac{\sum_{i=2}^n \lambda^{d_i}(1-\lambda^{d_{i-1}-d_i})(1-\lambda^{2m}\lambda^{c_i-d_i})}{(1-\lambda)^2}.\end{aligned}$$

(5) $\ell(S/V_m) - \ell(S/I) = m(2a+m)$.

Proof. To show that V_m is $\mathfrak{n}$-primary, we observe that $y^m f$ is not in any minimal prime P of $x^m(f+g)$. Since $a < b$, $X^m f^*$ is the leading form of $x^m(f+g)$. Since $y^* \nmid f^*$, we see that $y \notin P$. Since $(f, f+g)S = I$ and $x^* \nmid f^*$, we see that $f \notin P$. Thus V_m is $\mathfrak{n}$-primary. Let $V_m' = (f+g, y^m f)S$. The part (2) of [GHK2, Corollary 2.5] implies that $\mu_G(I) = \mu_G(V'_m)$ and the part (3) of [GHK2, Corollary 2.5] implies $\mu_G(V'_m)+1 = \mu_G(V_m)$. It also follows from Corollary 2.5 in [GHK2] that $Y^m\xi_1, X^m\xi_1, X^mY^m\xi_2, X^mY^m\xi_3, \dots, X^mY^m\xi_n$ is a minimal homogeneous system of generators for V_m^*. The remaining assertions in Theorem 2.2 follow from this. $\square$

In Theorem 2.3, we establish the existence of examples to show that every Hilbert series described in Theorem 1.3 is realizable as the Hilbert series of $\mathrm{gr}_{\mathfrak{m}}(R)$ for some complete intersection ideal $I = (f, g)$.

Theorem 2.3. *Let a and b be positive integers with $a \le b$ and consider a system consisting of*

(1) *An integer n with $2 \le n \le a+1$.*
(2) *A sequence $(a = c_1, b = c_2, c_3, \dots, c_n)$ of integers.*
(3) *A sequence $(a = d_1, d_2, \dots, d_n = 0)$ of integers such that* $c_{i+1} - c_i > d_{i-1} - d_i > 0$ *for all i with $2 \le i \le n-1$.*

For each system satisfying these conditions, there exists an ideal $I = (f, g)$ as in Setting 1.1 such that $(\xi_1 = f^*, \xi_2 = g^*, \xi_3, \dots, \xi_n)$ is a minimal set of homogeneous generators of I^*, $\deg \xi_i = c_i$, and $d_i = \deg D_i$, where $D_i = \mathrm{GCD}(\xi_1, \dots, \xi_i)$, for each i with $1 \le i \le n$.

Proof. The proof is by induction on a. If $a = 1$, then $n = 2$ and $f = x^a, g = y^b$ shows the assertion holds in this case. Let $a' > 1$ be an integer and assume that the assertion holds for all positive integers $a < a'$. Let $(a' = c'_1, b' = c'_2, c'_3, \dots, c'_{n+1})$ and $(a' = d'_1, d'_2, \dots, d'_{n+1} = 0)$ be sequences of $n+1$ integers such that

$$c'_{i+1} - c'_i > d'_{i-1} - d'_i > 0 \quad \text{for all} \quad i \quad \text{with} \quad 2 \le i \le n.$$

Notice that $d'_{i-1} - d'_i > 0$ for all i with $2 \le i \le n$ and $a' = \sum_{i=2}^{n+1}(d'_{i-1} - d'_i)$ implies $n+1 \le a'+1$. Let $e := c'_2 - c'_1$. By Proposition 2.1, the system consisting of the integer $n+1$ and the sequences $(c'_1, c'_2, \dots, c'_{n+1})$ and $(a' = d'_1, d'_2, \dots, d'_{n+1} = 0)$ is realizable if and only if the system consisting of $n+1$ and the sequences $(c'_1, c'_2 - e, \dots, c'_{n+1} - e)$ and $(a' = d'_1, d'_2, \dots, d'_{n+1} = 0)$ is realizable. Thus we may assume that $c'_1 = c'_2 = a'$.

Let $m = d'_1 - d'_2$ and $a = a' - m$. Then $a = d'_2$. Consider the system consisting of the positive integer n and the sequences $(a = c_1, c_2, \dots, c_n)$ and $(a = d_1, \dots, d_n)$, where $c_i = c'_{i+1} - 2m$ and $d_i = d'_{i+1}$ for each i with $2 \le i \le n$. We have

$$c_{i+1} - c_i > d_{i-1} - d_i > 0 \quad \text{for all} \quad i \quad \text{with} \quad 2 \le i \le n-1.$$

Also $a = \sum_{i=2}^{n}(d_{i-1} - d_i)$ implies $n \le a + 1$. By our inductive hypothesis the system consisting of the integer n and the sequences $(a = c_1, c_2, \dots, c_n)$ and $(a = d_1, \dots, d_n = 0)$ is realizable. Moreover, $c_2 = c_3'$ and $c_3' - a' > m$ and $a' = a + m$ implies $c_2 > a$. Therefore by Theorem 2.2 the system consisting of $n+1$ and the sequences $(c_1', c_2' - e, \dots, c_{n+1}' - e)$ and $(a' = d_1', d_2', \dots, d_{n+1}' = 0)$ is realizable. This completes the proof of Theorem 2.3. $\square$

3. Remarks and Examples

Remark 3.1. With notation and hypothesis as in Theorems 1.2 and 1.3, it follows from Theorem 1.3 that the Hilbert series $\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda) = \sum_{i=0}^{\infty} \dim_k [\mathrm{gr}_{\mathfrak{m}}(R)]_i \lambda^i$ is uniquely determined by the degree sequence $(c_1, c_2, \dots, c_n)$ together with the sequence $(a = d_1 > d_2 > d_3 > \cdots > d_n = 0)$. Notice also that $\dim_k [\mathrm{gr}_{\mathfrak{m}}(R)]_{c_i} = d_{i-1} - 1$, for each i with $2 \le i \le n$. Usually the sequence $(c_1, c_2, \cdots, c_n)$ does not uniquely determine the Hilbert series of $\mathrm{gr}_{\mathfrak{m}}(R)$, but Theorem 1.3 implies the following.

(1) $n = \mu_G(I^*) \le d_{n-1} + (n-1) \le \cdots \le d_3 + 3 \le d_2 + 2 \le a + 1$.

(2) $n = \mu_G(I^*) = a + 1 \iff d_{i-1} - d_i = 1$ for each i with $2 \le i \le a+1$. Therefore, in this case, $d_i = n - i$ for each i with $1 \le i \le n$, and the Hilbert series of $\mathrm{gr}_{\mathfrak{m}}(R)$ is uniquely determined by the sequence $(c_1, c_2, \cdots, c_n)$.

(3) If $\mu_G(I^*) = a + 1$, then $e(\mathrm{gr}_{\mathfrak{m}}(R)) \ge \frac{3a^2 - a}{2}$, and equality holds $\iff a = b$ and $c_{i+1} - c_i = 2$ for each i with $2 \le i \le a$. In this case, the degree sequence of I^* is $(a, a, a+2, a+4, a+6, \cdots, a + (a-1)2)$.

Example 3.2. We describe examples of parameter ideals $I = (f, g)S$ for which $\mu_G(I^*) = 3$. Let (a, b, c) be a sequence of integers such that $2 \le a \le b < b+2 \le c$. If (a, b, c) is the degree sequence of I^*, where $I = (f, g)$, then Theorem 1.3 implies that $d := d_2 = \deg \mathrm{GCD}(f^*, g^*)$ must satisfy $c - b > a - d > 0$.

Let
$$f = x^d y^{a-d} \qquad \text{and} \qquad g = x^b + y^{c+d-a}.$$
Notice that $c + d - a > b$. We have
$$I^* = (x^{*d} y^{*a-d}, x^{*b}, y^{*c}) = I_2 \begin{pmatrix} x^{*b-d} & y^{*a-d} & 0 \\ y^{*c+d-a} & 0 & x^{*d} \end{pmatrix}$$

Remark 3.3. Let $I = (f, g)S$ be a parameter ideal such that $\mu_G(I^*) = 3$. By Theorem 1.3, the Hilbert series of $\mathrm{gr}_{\mathfrak{m}}(R)$ is uniquely determined by the degree sequence (a, b, c) together with the integer $d := d_2 = \deg \mathrm{GCD}(f^*, g^*)$; moreover, we must have $c - b > a - d > 0$. Example 3.2 demonstrates that each system (a, b, c) and d with $c - b > a - d > 0$ is realizable. For integers (a, b, c) with $2 \leq a \leq b < b + 2 \leq c$, the possible values for d are constrained by the conditions $d \leq a - 1$ and $d > \min\{0, a + b - c\}$. Thus the number of possible values for d, and hence the number of different Hilbert series associated with the degree sequence (a, b, c), is $\min\{a - 1, c - b - 1\}$.

Example 3.4 illustrates Theorem 2.3 in the case where $\mu_G(I^*) = 4$.

Example 3.4. With the notation of Theorem 1.3, if I^* has degree sequence $(4, 5, 8, 11)$, then the possibilities for the sequence (d_1, d_2, d_3, d_4) are

$$(4, 3, 2, 0), \quad (4, 3, 1, 0), \quad (4, 2, 1, 0).$$

(1) This gives, for appropriate ideals I^*, the following three Hilbert series for G/I^*.

$$\begin{aligned}
\text{(i) } Q_{G/I^*}(t) &= 1 + 2t + 3t^2 + 4t^3 + 4t^4 + 3t^5 + 3t^6 + 3t^7 + 2t^8 + 2t^9 + 2t^{10} + t^{11}, \\
\text{(ii) } Q_{G/I^*}(t) &= 1 + 2t + 3t^2 + 4t^3 + 4t^4 + 3t^5 + 3t^6 + 3t^7 + 2t^8 + t^9 + t^{10}, \\
\text{(iii) } Q_{G/I^*}(t) &= 1 + 2t + 3t^2 + 4t^3 + 4t^4 + 3t^5 + 2t^6 + 2t^7 + t^8 + t^9 + t^{10}.
\end{aligned}$$

corresponding to the sequences $(4, 3, 2, 0)$, $(4, 3, 1, 0)$, $(4, 2, 1, 0)$.

(2) Using Proposition 2.1 and Theorem 2.2, we may obtain from the degree sequence $(4, 5, 8, 11)$ of I^*, either the degree sequence $(3, 3, 6)$ or $(2, 2, 5)$. Indeed, $(3, 3, 6)$ is obtained by

$$(4, 5, 8, 11) \xrightarrow{\text{m=1 in Prop 2.1}} (4, 4, 7, 10) \xrightarrow{\text{m=1 in Thm 2.2}} (3, 5, 8) \xrightarrow{\text{m=2 in Prop 2.1}} (3, 3, 6),$$

and $(2, 2, 5)$ is obtained by

$$(4, 5, 8, 11) \xrightarrow{\text{m=1 in Prop 2.1}} (4, 4, 7, 10) \xrightarrow{\text{m=2 in Thm 2.2}} (2, 3, 6) \xrightarrow{\text{m=1 in Prop 2.1}} (2, 2, 5).$$

The sequence $(3, 3, 6)$ is the degree sequences associated with $(x^3 + y^5, x^2y)S$ and $(2, 2, 5)$ is the degree sequence associated with $(x^2 + y^4, xy)S$.

(i) Let $I_1^{'} = (f_1, g_1)$, where $f_1 = x^3 + y^5$, $g_1 = x^2y$. Since $y^* \nmid f_1^*$, Proposition 2.1 implies that $J_1^{'} = (f_1, y^2g_1)$ has degree sequence $(3, 5, 8)$. Let $\mathbf{n} = (x + y, y)$. Then $(x^* + y^*) \nmid f_1^*$ and $y^* \nmid f_1^*$, so Theorem 2.2 implies that $V_1^{'} = (yf_1, (x+y)(f_1+y^2g_1)) = (F_1, G_1)$ has degree sequence $(4, 4, 7, 10)$. Since $y^* \nmid G_1^*$, Proposition 2.1 implies that $I_1 = (yF_1, G_1) = (y^2f_1, (x + y)(f_1 + y^2g_1))$ has degree sequence $(4, 5, 8, 11)$. Also the descending sequence (d_1, d_2, d_3, d_4) of I_1^* is $(4, 3, 2, 0)$. Hence the Hilbert series of G/I_1^* is (i) of part (1).

(ii) Let $I_2^{'} = (f_2, g_2)$, where $f_2 = x^3 + y^4$, $g_2 = xy^2$. Since $y^* \nmid f_2^*$, Proposition 2.1 implies that $J_2^{'} = (f_2, y^2 g_2)$ has degree sequence $(3, 5, 8)$. Let $\mathbf{n} = (x + y, y)$. Then $(x^* + y^*) \nmid f_2^*$ and $y^* \nmid f_2^*$, so Theorem 2.2 implies that $V_2^{'} = (yf_2, (x+y)(f_2+y^2g_2)) = (F_2, G_2)$ has degree sequence $(4, 4, 7, 10)$. Since $y^* \nmid G_2^*$, Proposition 2.1 implies that $I_2 = (yF_2, G_2) = (y^2 f_2, (x + y)(f_2 + y^2 g_2)$ has degree sequence $(4, 5, 8, 11)$. The descending sequence (d_1, d_2, d_3, d_4) of I_2^* is $(4, 3, 1, 0)$. Hence the Hilbert series of G/I_2^* is (ii) of part (1).

(iii) Let $I_3^{'} = (f_3, g_3)$, where $f_3 = x^2 + y^4$, $g_3 = xy$. Since $y^* \nmid f_3^*$, Proposition 2.1 implies that $J_3^{'} = (f_3, yg_3)$ has degree sequence $(2, 3, 6)$. Let $\mathbf{n} = (x + y, y)$. Then $(x^* + y^*) \nmid f_3^*$ and $y^* \nmid f_3^*$, so Theorem 2.2 implies that $V_3^{'} = (y^2 f_3, (x + y)^2(f_3 + yg_3)) = (F_3, G_3)$ has degree sequence $(4, 4, 7, 10)$. Since $y^* \nmid G_3^*$, Proposition 2.1 implies that $I_3 = (yF_3, G_3) = (y^3 f_3, (x+y)^2(f_3+yg_3)$ has degree sequence $(4, 5, 8, 11)$. The descending sequence (d_1, d_2, d_3, d_4) of I_3^* is $(4, 2, 1, 0)$. Hence the Hilbert series of G/I_3^* is (iii) of part (1).

Example 3.5 illustrates Theorem 2.3 in the case where $\mu_G(I^*) = 5$.

Example 3.5. Assume notation as in Theorems 1.2 and 1.3 and also assume that $\mathrm{char}(k) \neq 2$. If I^* has degree sequence $(6, 7, 12, 15, 18)$, then the possibilities for the sequence $(d_1, d_2, d_3, d_4, d_5)$ are

$$(6,5,4,3,0), \quad (6,5,4,2,0), \quad (6,5,3,2,0), \quad (6,5,3,1,0),$$
$$(6,4,3,2,0), \quad (6,4,3,1,0), \quad (6,4,2,1,0), \quad (6,3,2,1,0).$$

The following eight parameter ideals $I_i = (f_i, g_i)S$ have associated degree sequence $(6, 7, 12, 15, 18)$ and each of the eight possible sequences $(d_1, d_2, d_3, d_4, d_5)$.

(i) $I_1 = (yy(x + y)(f_1 + y^2 g_1), \quad (x + 2y)(x + y)(f_1 + y^2 g_1 + y^4 y f_1))$
where $f_1 = x^4 + y^6$ and $g_1 = x^3 y$,

(ii) $I_2 = (yy(x + y)(f_2 + y^2 g_2), \quad (x + 2y)(x + y)(f_2 + y^2 g_2 + y^4 y f_2))$,
where $f_2 = x^4 + y^5$ and $g_2 = x^2 y^2$,

(iii) $I_3 = (yy(x + y)^2(f_3 + yg_3), \quad (x + 2y)(x + y)^2(f_3 + yg_3 + y^4 y^2 f_3))$,
where $f_3 = x^3 + y^5$ and $g_3 = x^2 y$,

(iv) $I_4 = (yy^2(x+y)(f_4 + y^2g_4), \quad (x+2y)^2(x+y)(f_4 + y^2g_4 + y^3yf_4)),$
where $\quad f_4 = x^3 + y^5 \quad$ and $\quad g_4 = x^2y,$

(v) $I_5 = (yy(x+y)^2(f_5 + yg_5), \quad (x+2y)(x+y)^2(f_5 + y^2g_5 + y^4y^2f_5)),$
where $\quad f_5 = x^3 + y^4 \quad$ and $\quad g_5 = xy^2,$

(vi) $I_6 = (yy^2(x+y)(f_6 + y^2g_6), \quad (x+2y)^2(x+y)(f_6 + y^2g_6 + y^3yf_6)),$
where $\quad f_6 = x^3 + y^4 \quad$ and $\quad g_6 = xy^2,$

(vii) $I_7 = (yy^2(x+y)^2(f_7 + yg_7), \quad (x+2y)^2(x+y)^2(f_7 + yg_7 + y^3yf_7)),$
where $\quad f_7 = x^2 + y^4 \quad$ and $\quad g_7 = xy,$

(viii) $I_8 = (yy^3(x+y)(f_8 + y^2g_8), \quad (x+2y)^3(x+y)(f_8 + y^2g_8 + y^2yf_8)),$
where $\quad f_8 = x^2 + y^4 \quad$ and $\quad g_8 = xy.$

To obtain these parameter ideals we reason as follows. Using Proposition 2.1 and Theorem 2.2, we may obtain from the degree sequence $(6, 7, 12, 15, 18)$ of I^* one of the following three degree sequences $(4, 4, 7)$, $(3, 3, 6)$, or $(2, 2, 5)$. Indeed, $(4, 4, 7)$ is obtained by

$$(6,7,12,15,18) \overset{\text{m=1 in Prop 2.1}}{\longrightarrow} (6,6,11,14,17) \overset{\text{m=1 in Thm 2.2}}{\longrightarrow} (5,9,12,15)$$
$$\overset{\text{m=4 in Prop 2.1}}{\longrightarrow} (5,5,8,11) \overset{\text{m=1 in Thm 2.2}}{\longrightarrow} (4,6,9) \overset{\text{m=2 in Prop 2.1}}{\longrightarrow} (4,4,7).$$

The sequence $(3, 3, 6)$ is obtained in two ways:

$$(6,7,12,15,18) \overset{\text{m=1 in Prop 2.1}}{\longrightarrow} (6,6,11,14,17) \overset{\text{m=1 in Thm 2.2}}{\longrightarrow} (5,9,12,15)$$
$$\overset{\text{m=4 in Prop 2.1}}{\longrightarrow} (5,5,8,11) \overset{\text{m=2 in Thm 2.2}}{\longrightarrow} (3,4,7) \overset{\text{m=1 in Prop 2.1}}{\longrightarrow} (3,3,6).$$

and

$$(6,7,12,15,18) \overset{\text{m=1 in Prop 2.1}}{\longrightarrow} (6,6,11,14,17) \overset{\text{m=2 in Thm 2.2}}{\longrightarrow} (4,7,10,13)$$
$$\overset{\text{m=3 in Prop 2.1}}{\longrightarrow} (4,4,7,10) \overset{\text{m=1 in Thm 2.2}}{\longrightarrow} (3,5,8) \overset{\text{m=2 in Prop 2.1}}{\longrightarrow} (3,3,6).$$

The sequence $(2, 2, 5)$ is also obtained in two ways:

$$(6,7,12,15,18) \overset{\text{m=1 in Prop 2.1}}{\longrightarrow} (6,6,11,14,17) \overset{\text{m=2 in Thm 2.2}}{\longrightarrow} (4,7,10,13)$$
$$\overset{\text{m=3 in Prop 2.1}}{\longrightarrow} (4,4,7,10) \overset{\text{m=2 in Thm 2.2}}{\longrightarrow} (2,3,6) \overset{\text{m=1 in Prop2.1}}{\longrightarrow} (2,2,5).$$

and

$$(6,7,12,15,18) \overset{\text{m=1 in Prop 2.1}}{\longrightarrow} (6,6,11,14,17) \overset{\text{m=3 in Thm 2.2}}{\longrightarrow} (3,5,8,11)$$
$$\overset{\text{m=2 in Prop 2.1}}{\longrightarrow} (3,3,6,9) \overset{\text{m=1 in Thm 2.2}}{\longrightarrow} (2,4,7) \overset{\text{m=2 in Prop 2.1}}{\longrightarrow} (2,2,5).$$

We describe a procedure for obtaining the parameter ideal I_8. Let $I_8^{''} = (f_8, g_8)$, where $f_8 = x^2 + y^4$, $g_8 = xy$. Then $I_8^{''}$ has degree sequence $(2,2,5)$. Since $y^* \nmid f_8^*$, Proposition 2.1 implies that $J_8^{''} = (f_8, y^2 g_8)$ has degree sequence $(2,4,7)$. Let $\mathbf{n} = (x+y, y)$. Then $(x^* + y^*) \nmid f_8^*$ and $y^* \nmid f_8^*$, so Theorem 2.2 implies that $V_8^{''} = (yf_8, (x+y)(f_8 + y^2 g_8)) = (F_8^{'}, G_8^{'})$ has degree sequence $(3,3,6,9)$. Since $y^* \nmid G_8^{'*}$, Proposition 2.1 implies that $J_8^{'} = (y^2 F_8^{'}, G_8^{'})$ has degree sequence $(3,5,8,11)$. Since $\mathbf{n} = (x+2y, y) = (x+y, y) = (x,y)$ and $(x+2y)^* \nmid G_8^{'*}$ and $y^* \nmid G_8^{'*}$, Theorem 2.2 implies that $V_8^{'} = (y^3 G_8^{'}, (x+2y)^3 (G_8^{'} + y^2 F_8^{'})) = (F_8, G_8)$ has degree sequence $(6,6,11,14,17)$. Since $y^* \nmid G_8^*$, Proposition 2.1 implies that $I_8 = (yF_8, G_8) = (y^4 G_8^{'}, (x+2y)^3 (G_8^{'} + y^2 F_8^{'}) = (y^4 (x+y)(f_8 + y^2 g_8), (x+2y)^3 (x+y)(f_8 + y^2 g_8 + y^3 f_8))$, has degree sequence $(6,7,12,15,18)$.

Similar reasoning is used to obtain the parameter ideals I_i, for $1 \leq i \leq 7$.

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Department of Mathematics, School of Science and Technology Meiji University, 214-8571 Japan

E-mail address: goto@math.meiji.ac.jp

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907

E-mail address: heinzer@math.purdue.edu

Department of Mathematics, Sungkyunkwan University Jangangu Suwon 440-746, Korea

E-mail address: mkkim@skku.edu