EXAMPLES OF INTEGRAL DOMAINS INSIDE POWER SERIES RINGS

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Abstract. We present examples of Noetherian and non-Noetherian integral domains which can be built inside power series rings. Given a power series ring R^* over a Noetherian integral domain R and given a subfield L of the total quotient ring of R^* with $R \subseteq L$, we construct subrings A and B of L such that B is a localization of a nested union of polynomial rings over R and $B \subseteq A := L \cap R^*$. We show in certain cases that flatness of a related map on polynomial rings is equivalent to the Noetherian property for B. Moreover if B is Noetherian, then $B = A$. We use this construction to obtain for each positive integer n an explicit example of a 3-dimensional quasilocal unique factorization domain B such that the maximal ideal of B is 2-generated, B has precisely n prime ideals of height two, and each prime ideal of B of height two is not finitely generated.

1. Introduction. This paper is a continuation of our study of a technique for constructing integral domains by (1) intersecting a power series ring with a field to obtain an integral domain A as in the abstract, and (2) approximating the domain A with a nested union of localized polynomial rings to obtain an integral domain B as in the abstract. Classical examples such as those of Akizuki [A] and Nagata [N, pages 209-211] use the second (nested union) description of this construction. It is possible to also realize these classical examples as the intersection domains of the first description [HRW6].

In this paper we observe that, in certain applications of this technique, flatness of a map of associated polynomial rings implies the constructed domains are Noetherian and that $A = B$. We also in the present paper apply this observation to the construction of examples of both Noetherian and non-Noetherian integral domains.

We begin by describing the technique.

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1.1 General Setting. Let R be a commutative Noetherian integral domain. Let a be a nonzero nonunit of R and let R^* be the (a)-adic completion of R. Then R^* is isomorphic to $R[[y]]/(y-a)$, where y is an indeterminate; thus we consider R^* as $R[[a]]$, the "power series ring" in a over R. The intersection domain (type 1 above) and the approximation domain (type 2) of the construction are inside R^*. Let $\tau_1, \dots, \tau_n \in aR^*$ be algebraically independent over the fraction field K of R and let $\underline{\tau}$ abbreviate the list $\tau_1, \dots, \tau_n$. By Theorem 2.2, also known as [HRW1, Theorem 1.1], $A_{\underline{\tau}} := K(\tau_1, \dots, \tau_n) \cap R^*$ is simultaneously Noetherian and computable as a nested union $B_{\underline{\tau}}$ of certain associated localized polynomial rings over R using $\underline{\tau}$ if and only if the extension $T := R[\underline{\tau}] := R[\tau_1, \dots, \tau_n] \overset{\psi}{\hookrightarrow} R^*_a$ is flat.

In the case where $\psi : T \hookrightarrow R^*_a$ is flat, so that the intersection domain $A_{\underline{\tau}}$ is Noetherian and computable, we construct new "insider" examples inside $A_{\underline{\tau}}$. We choose elements $f_1, \dots, f_m$ of T, considered as polynomials in the τ_i with coefficients in R and abbreviated by $\underline{f}$. Assume that $f_1, \dots, f_m$ are algebraically independent over K; thus $m \le n$. If $S := R[\underline{f}] := R[f_1, \dots, f_m] \overset{\varphi}{\hookrightarrow} T = R[\underline{\tau}]$ is flat, we observe in Section 3 that the "insider ring" $A_{\underline{f}} := K(\underline{f}) \cap R^*$ is Noetherian and computable; that is, $A_{\underline{f}}$ is equal to an approximating union $B_{\underline{f}}$ of localized polynomial rings constructed using the f_i. Moreover, we can often identify conditions on the map φ which imply $B_{\underline{f}}$ and $A_{\underline{f}}$ are not Noetherian. Thus the "insider" examples $A_{\underline{f}}$ and $B_{\underline{f}}$ are inside intersection domains $A_{\underline{\tau}}$ known to be Noetherian; the new insider is Noetherian if the associated extension $S \to T$ of polynomial rings is flat. The insider examples are examined in more detail in Section 3.

In Section 2 we give background and notation for the construction and for flatness of polynomial extensions in greater generality: Suppose that $\underline{x} := (x_1, \dots, x_n)$ is a tuple of indeterminates over R and that $\underline{f} := (f_1, \dots, f_m)$ consists of elements of the polynomial ring $R[\underline{x}]$ that are algebraically independent over K. We consider flatness of the following map of polynomial rings.

$$\varphi : S := R[\underline{f}] \hookrightarrow T := R[\underline{x}]. \tag{1.2}$$

In Section 4 we continue the analysis of the flatness of (1.2) and the nonflat locus. We discuss results of [P], [W] and others.

In Section 5 we present for each positive integer n an insider example B such that:

(1) B is a 3-dimensional quasilocal unique factorization domain,
(2) B is not catenary,
(3) the maximal ideal of B is 2-generated,
(4) B has precisely n prime ideals of height two,
(5) Each prime ideal of B of height two is not finitely generated,
(6) For every non-maximal prime P of B the ring B_P is Noetherian.

2. Background and Notation.

We begin this section by recalling some details for the approximation to the intersection domain A_τ of (1.1).

2.1 Notation for approximations. Assume that R, K, a, $\tau_1, \dots, \tau_n$, $\underline{\tau}$ and $A_{\underline{\tau}}$ are as in General Setting 1.1. Then the (a)-adic completion of R is $R^* = R[[x]]/(x-a) = R[[a]]$. Write each $\tau_i := \sum_{j=1}^{\infty} b_{ij}a^j$, with the $b_{ij} \in R$. There are natural sequences $\{\tau_{ir}\}_{r=0}^{\infty}$ of elements in A, called the r^{th} *endpieces* for the τ_i, which "approximate" the τ_i, defined by:

$$\text{(2.1.1)} \qquad \text{For each } i \in \{1, \dots, n\} \text{ and } r \geq 0, \quad \tau_{ir} := \sum_{j=r+1}^{\infty} (b_{ij}a^j)/a^r.$$

Now for each r, $U_r := R[\tau_{1r}, \dots, \tau_{nr}]$ and B_r is U_r localized at the multiplicative system $1+aU_r$. Then define $U_{\underline{\tau}} := \cup_{r=1}^{\infty} U_r$ and $B_{\underline{\tau}} := \cup_{r=1}^{\infty} B_r$. Thus $U_{\underline{\tau}}$ is a nested union of polynomial rings over R and $B_{\underline{\tau}}$ is a nested union of localized polynomial rings over R. The definition of the U_r (and hence also of B_r and $U_{\underline{\tau}}$ and $B_{\underline{\tau}}$) are independent of the representation of the τ_i as power series with coefficients in R [HRW1, Proposition 2.3].

The following theorem is the basis for our construction of examples.

2.2 Theorem. [HRW1, Theorem 1.1] *Let R be a Noetherian integral domain with fraction field K. Let a be a nonzero nonunit of R. Let $\tau_1, \dots, \tau_n \in aR[[a]] = aR^*$ be algebraically independent over K, abbreviated by $\underline{\tau}$. Let $U_{\underline{\tau}}$ and $B_{\underline{\tau}}$ be as in (2.1). Then the following statements are equivalent:*

(1) *$A_{\underline{\tau}} := K(\underline{\tau}) \cap R^*$ is Noetherian and $A_{\underline{\tau}} = B_{\underline{\tau}}$.*
(2) *$U_{\underline{\tau}}$ is Noetherian.*

(3) $B_{\underline{\tau}}$ *is Noetherian.*

(4) $R[\underline{\tau}] \to R_a^*$ *is flat.*

Since flatness is a local property, the following two propositions are immediate corollaries of [HRW5, Theorem 2.1]; see also [P, Théorème 3.15].

2.3 Proposition. *Let T be a Noetherian ring and suppose $R \subseteq S$ are Noetherian subrings of T. Assume that $R \to T$ is flat with Cohen-Macaulay fibers and that $R \to S$ is flat with regular fibers. Then $S \to T$ is flat if and only if, for each prime ideal P of T, we have $ht(P) \geq ht(P \cap S)$.*

As a special case we have:

2.4 Proposition. *Let R be a Noetherian ring and let $x_1, \ldots, x_n$ be indeterminates over R. Assume that $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$ are algebraically independent over R. Then*

(1) *$\varphi : S := R[f_1, \ldots, f_m] \hookrightarrow T := R[x_1, \ldots, x_n]$ is flat if and only if, for each prime ideal P of T, we have $ht(P) \geq ht(P \cap S)$.*

(2) *For $Q \in \operatorname{Spec} T$, $\varphi_Q : S \to T_Q$ is flat if and only if for each prime ideal $P \subseteq Q$ of T, we have $ht(P) \geq ht(P \cap S)$.*

2.5 Definitions and Remarks. (1) The *Jacobian ideal* J of the extension (1.2) is the ideal generated by the $m \times m$ minors of the $m \times n$ matrix $\mathcal{J}$ given below:

$$\mathcal{J} := \left(\frac{\partial f_i}{\partial x_j} \right)_{i,j}.$$

(2) For the extension (1.2), the *nonflat locus* of φ is the set $\mathcal{F}$, where

$$\mathcal{F} := \{ Q \in \operatorname{Spec}(T) : \text{the map} \quad \varphi_Q : S \to T_Q \text{ is not flat } \}.$$

For convenience, we also define the set $\mathcal{F}_{\min}$ and the ideal F of T:

$$\mathcal{F}_{\min} := \{ \text{ minimal elements of } \mathcal{F} \} \text{ and } F := \cap \{ Q : Q \in \mathcal{F} \}.$$

By [M2, Theorem 24.3], the set $\mathcal{F}$ is closed in the Zariski topology and hence is equal to $\mathcal{V}(F)$, the set of primes of T that contain the ideal F. Thus the set $\mathcal{F}_{\min}$ is a finite set and consists precisely of the minimal primes of the ideal F.

Moreover, Proposition 2.4 implies $\mathcal{F}_{\min} \subseteq \{ Q \in \operatorname{Spec} T : \operatorname{ht} Q < \operatorname{ht}(Q \cap S) \}$ and for every prime ideal $P \subsetneq Q \in \mathcal{F}_{\min}$, $\operatorname{ht} P \geq \operatorname{ht}(P \cap S)$.

(3) In general for a commutative ring T and a subring R, we say that elements $f_1, \dots, f_m \in T$ are *algebraically independent* over R if, for indeterminates $t_1, \dots, t_m$ over R, the only polynomial $G(t_1, \dots, t_m) \in R[t_1, \dots, t_m]$ with $G(f_1, \dots, f_m) = 0$ is the zero polynomial.

2.6 Example and Remarks. (1) Let k be a field, let x and y be indeterminates over k and set $f = x$, $g = (x-1)y$. Then $k[f,g] \xrightarrow{\varphi} k[x,y]$ is not flat.

Proof. For the prime ideal $P := (x-1) \in \operatorname{Spec}(k[x,y])$, we see that $\operatorname{ht}(P) = 1$, but $\operatorname{ht}(P \cap k[f,g]) = 2$; thus the extension is not flat by Proposition 2.4.

(2) The Jacobian ideal J of f and g in (1) is given by:

$$J = (\det \begin{pmatrix} \frac{\delta f}{\delta x} & \frac{\delta f}{\delta y} \\ \frac{\delta g}{\delta x} & \frac{\delta g}{\delta y} \end{pmatrix}) = (\det \begin{pmatrix} 1 & 0 \\ y & x-1 \end{pmatrix}) = (x-1).$$

(3) In this example the nonflat locus is equal to the set of prime ideals Q of $k[x,y]$ which contain the Jacobian ideal $(x-1)k[x,y]$, thus $J = F$.

We record in Proposition 2.7 observations about flatness that follow from well-known properties of the Jacobian.

2.7 Proposition. *Let R be a Noetherian ring, let $x_1, \dots, x_n$ be indeterminates over R, and let $f_1, \dots, f_m \in R[x_1, \dots, x_n]$ be algebraically independent over R. Consider the embedding $\varphi : S := R[f_1, \dots, f_m] \hookrightarrow T := R[x_1, \dots, x_n]$. Let J denote the Jacobian ideal of φ and let $Q \in \operatorname{Spec} T$. Then*

(1) *Q does not contain J if and only if $\varphi_Q : S \to T_Q$ is essentially smooth.*
(2) *If Q does not contain J, then $\varphi_Q : S \to T_Q$ is flat. Thus $J \subseteq F$.*
(3) *$\mathcal{F}_{\min} \subseteq \{Q' \in \operatorname{Spec} T : J \subseteq Q'$ and $\operatorname{ht}(Q' \cap S) > \operatorname{ht} Q'\}$.*

Proof. For item 1, we observe that our definition of the Jacobian ideal J given in (2.5) agrees with the description of the smooth locus of an extension given in [E], [S, Section 4].

To see this, let $u_1, \dots, u_m$ be indeterminates over $R[x_1, \dots, x_n]$ and identify

$$R[x_1, \dots, x_n] \quad \text{with} \quad \frac{R[u_1, \dots, u_m][x_1, \dots, x_n]}{(\{u_i - f_i\}_{i=1,\dots,m})}.$$

Since $u_1, \dots, u_m$ are algebraically independent, the ideal J generated by the minors of $\mathcal{J}$ is the Jacobian ideal of the extension (1.2) by means of this identification. We make this more explicit as follows.

Let $U_1 := R[u_1, \dots, u_m, x_1, \dots, x_n]$ and $I = (\{f_i - u_i\}_{i=1,\dots,m})U_1$. Consider the following commutative diagram

$$\begin{array}{ccc} S := R[f_1, \dots, f_m] & \longrightarrow & T := R[x_1, \dots, x_n] \\ \cong\big\downarrow & & \cong\big\downarrow \\ S_1 := R[u_1, \dots, u_m] & \longrightarrow & T_1 := R[u_1, \dots, u_m, x_1, \dots, x_n]/I \end{array}$$

Define as in [E], [S, Section 4]

$$H = H_{T_1/S_1} := \text{the radical of } \Sigma\Delta(g_1, \dots, g_s)[(g_1, \dots, g_s) : I],$$

where the sum is taken over all s with $0 \le s \le m$, for all choices of s polynomials $g, \dots, g_s$ from $I = (\{f_1 - u_1, \dots, f_m - u_m\})U_1$, where $\Delta := \Delta(g_1, \dots, g_s)$ is the ideal of $T \cong T_1$ generated by the $s \times s$-minors of $\left(\frac{\partial g_i}{\partial x_j}\right)$, and $\Delta = T$ if $s = 0$.

To establish (2.7.1), we show that $H = \operatorname{rad}(J)$. Since u_i is a constant with respect to x_j, we have $\left(\frac{\partial (f_i - u_i)}{\partial x_j}\right) = \left(\frac{\partial f_i}{\partial x_j}\right)$. Thus $J \subseteq H$.

For $g_1, \dots, g_s \in I$, the $s \times s$-minors of $\left(\frac{\partial g_i}{\partial x_j}\right)$ are contained in the $s \times s$-minors of $\left(\frac{\partial f_i}{\partial x_j}\right)$. Thus it suffices to consider s polynomials $g_1, \dots, g_s$ from the set $\{f_1 - u_1, \dots, f_m - u_m\}$. Now $f_1 - u_1, \dots, f_m - u_m$ is a regular sequence in $R[u_1 \dots u_r, x_1, \dots, x_n]$. Thus for $s < m$, $[(g_1, \dots, g_s) : I] = (g_1, \dots, g_s)$. Thus the $m \times m$-minors of $\left(\frac{\partial f_i}{\partial x_j}\right)$ generate H up to radical, and so $H = \operatorname{rad}(J)$.

Hence by [E] or [S, Theorem 4.1], T_Q is essentially smooth over S if and only if Q does not contain J.

Item 2 follows from item 1 because essentially smooth maps are flat. In view of Proposition 2.4 and (2.5.2), item 3 follows from item 2. □

2.8 Remarks. (1) For φ as in (1.2), it would be interesting to identify the set $\mathcal{F}_{\min}$. In particular we are interested in conditions for $J = F$ and/or conditions for $J \subsetneq F$.

(2) If $\operatorname{char} R = 0$, then the zero ideal is not in $\mathcal{F}_{\min}$ and so $F \neq \{0\}$.

(3) In view of (2.7.3), we can describe $\mathcal{F}_{\min}$ exactly as

$\mathcal{F}_{\min} = \{Q \in \operatorname{Spec} T : J \subseteq Q, \operatorname{ht}(Q \cap S) > \operatorname{ht} Q \text{ and } \forall P \subsetneq Q, \operatorname{ht}(P) \ge \operatorname{ht}(P \cap S)\}$.

(4) By (2.8.3), every prime ideal Q of $\mathcal{F}_{\min}$ contains two primes $P_1 \subsetneq P_2$ of S such that Q is minimal above both P_1T and P_2T.

3. Explicit constructions inside simpler extensions.

Using Theorem 2.2 and intersection domains inside the completion which are known to be Noetherian, we formulate a shortcut method for the construction of "insider" examples.

3.1 General Method. Let R be a Noetherian integral domain. Let a be a nonzero nonunit of R and let $R^* = R[[x]]/(x-a)$ be the (a)-adic completion of R. Let $\tau_1, \dots, \tau_n \in aR^*$, abbreviated by $\underline{\tau}$, be algebraically independent over the fraction field K of R. Assume that the extension $T := R[\tau_1, \dots, \tau_n] \overset{\psi}{\hookrightarrow} R^*_a$ is flat. Thus by Theorem 2.2, $D := A_{\underline{\tau}} = K(\tau_1, \dots, \tau_n) \cap R^*$ is Noetherian and computable as a nested union of localized polynomial rings over R using the τ's.

Let $f_1, \dots, f_m$ be elements of T, abbreviated by $\underline{f}$ and considered as polynomials in the τ_i with coefficients in R. Assume that $f_1, \dots, f_m$ are algebraically independent over K; thus $m \le n$. Let $S := R[\underline{f}] \overset{\varphi}{\hookrightarrow} T = R[\underline{\tau}]$; put $\alpha := \psi \circ \varphi : S \to R^*_a$. That is, we have:

$$\begin{tikzcd} & & R^*_a \\ R \subseteq S := R[\underline{f}] \arrow[urr, "\alpha:=\psi\varphi"] \arrow[rr, "\varphi"] & & T := R[\underline{\tau}] \arrow[u, "\psi"'] \end{tikzcd}$$

Using the f's in place of the τ's, we define the ring $A := A_{\underline{f}} := K(\underline{f}) \cap R^*$ and the approximation rings $U_r, B_r, U_{\underline{f}}$ and $B = B_{\underline{f}}$, as in (2.1). Let

$$F := \cap \{ P \in \mathrm{Spec}(T) \mid \varphi_P : S \to T_P \text{ is not flat } \}.$$

Thus, as in (2.5.2), the ideal F defines the nonflat locus of the map $\varphi : S \to T$. For $Q^* \in \mathrm{Spec}(R^*_a)$, we consider whether the localized map $\varphi_{Q^* \cap T}$ is flat:

(3.1.1) $$\varphi_{Q^* \cap T} : S \to T_{Q^* \cap T}$$

3.2 Theorem. *With the notation of (3.1) we have*

1. *For $Q^* \in \mathrm{Spec}(R^*_a)$, the map $\alpha_{Q^*} : S \to (R^*_a)_{Q^*}$ is flat if and only if the map $\varphi_{Q^* \cap T}$ in (3.1.1) is flat.*
2. *The following are equivalent:*
   - *(i) A is Noetherian and $A = B$.*
   - *(ii) B is Noetherian.*
   - *(iii) The map $\varphi_{Q^* \cap T}$ in (3.1.1) is flat for every maximal $Q^* \in \mathrm{Spec}(R^*_a)$.*

(iv) $FR_a^* = R_a^*$.

(3) $\varphi_a : S \to T_a$ *is flat if and only if* $FT_a = T_a$. *Moreover, either of these conditions implies* B *is Noetherian and* $B = A$.

Proof. For item (1), we have $\alpha_{Q^*} = \psi_{Q^*} \circ \varphi_{Q^* \cap T} : S \to T_{Q^* \cap T} \to (R_a^*)_{Q^*}$. Since the map ψ_{Q^*} is faithfully flat, the composition α_{Q^*} is flat if and only if $\varphi_{Q^* \cap T}$ is flat [M1, page 27]. For item (2), the equivalence of (i) and (ii) is part of Theorem 2.2. The equivalence of (ii) and (iii) follows from item (1) and Theorem 2.2. For the equivalence of (iii) and (iv), we use $FR^* \neq R^* \iff F \subseteq Q^* \cap T$, for some Q^* maximal in $\operatorname{Spec}(R^*)_a) \iff$ the map in (3.1.1) fails to be flat. Item (3) follows from the definition of F and the fact that the nonflat locus of $\varphi : S \to T$ is closed. $\square$

To examine the map $\alpha : S \to R_a^*$ in more detail, we use the following terminology.

3.3 Definition. For an extension of Noetherian rings $\varphi : A' \hookrightarrow B'$ and for $d \in \mathbb{N}$, we say that $\varphi : A' \hookrightarrow B'$, *satisfies* LF_d if for each $P \in \operatorname{Spec}(B')$ with $\operatorname{ht}(P) \leq d$, the composite map $A' \to B' \to B'_P$ is flat.

3.4 Corollary. *With the notation of (3.1), we have* $\operatorname{ht}(FR_a^*) > 1 \iff \varphi : S \to R_a^*$ *satisfies* $LF_1 \iff B = A$.

Proof. The first equivalence follows from the definition of LF_1 and the second equivalence from [HRW4,Theorem 5.5].

3.5 A more concrete situation. Let $R := k[x, y_1, \dots, y_s]$, where k is a field and $x, y_1, \dots, y_s$ are indeterminates over k with the y_i abbreviated by $\underline{y}$. Let $R^* = k[\underline{y}][[x]]$, the (x)-adic completion of R. Let $\tau_1, \dots, \tau_n$, abbreviated by $\underline{\tau}$, be elements of $xk[[x]]$ which are algebraically independent over $k(x)$. Let $D := A_{\underline{\tau}} := k(x, \underline{y}, \underline{\tau}) \cap R^*$. Let $T = R[\underline{\tau}]$. Then $T \to R_x^*$ is flat, D is a nested union of localized polynomial rings obtained using the τ_i and D is a Noetherian regular local ring; moreover, if $\operatorname{char} k = 0$, then D is excellent [HRW3, Proposition 4.1].

We now use the procedure of (3.1) to construct examples inside D. Let $f_1, \dots, f_m$, abbreviated by $\underline{f}$, be elements of T considered as polynomials in $\tau_1, \dots, \tau_n$ with coefficients in R, that are algebraically independent over $k(x, \underline{y})$. We assume the constant terms in $R = k[x, \underline{y}]$ of the f_i are zero. Let $S := R[\underline{f}]$. The inclusion map $S \hookrightarrow T$ is an injective R-algebra homomorphism, and $m \leq n$.

Let $A := \mathcal{Q}(S) \cap R^*$ and let B be the nested union domain associated to the $\underline{f}$, as in (2.1). By Theorem 2.2, B is Noetherian and $B = A$ if and only if the map $\alpha : S \to R^*_x$ is flat. Furthermore, by Theorem 3.2, we can recover information about flatness of α by considering the map $\varphi : S \to T$.

The following remark describes how the f_i are chosen in several classical examples:

3.6 Remark. With the notation of (3.5).

(1) Nagata's famous example [N1], [N2, Example 7, page 209], [HRW6, Example 3.1], may be described by taking $n = s = m = 1, y_1 = y, \tau_1 = \tau$, and $f_1 = f$ and localizing. Then $R = k[x, y]_{(x,y)}$, $T = k[x, y, \tau]_{(x,y,\tau)}$, $f = (y + \tau)^2$, $S = k[x, y, f]_{(x,y,f)}$ and $A = k(x, y, (y + \tau)^2) \cap R^*$. The Noetherian property of B is implied by the flatness property of the map $S \to T_x$. Thus $B = A$. In this case, T is actually a free S-module with $< 1, y + \tau >$ as a free basis.

(2) An example of Rotthaus [R1],[HRW6, Example 3.3], may be described by taking $n = s = 2$, and $m = 1$ and localizing. Then $R = k[x, y_1, y_2]_{(x,y_1,y_2)}$, $T = R[\tau_1, \tau_2]_{(\mathbf{m},\tau_1,\tau_2)}$, $f_1 = (y_1 + \tau_1)(y_2 + \tau_2)$, $S = R[f_1]_{(\mathbf{m},f_1)}$ and $A = k(x, y_1, y_2, (y_1+\tau_1)(y_2+\tau_2)) \cap R^*$. Since the map from $R[f_1] \to R_x[\tau_1, \tau_2] = T_x$ is flat, the associated nested union domain B is Noetherian.

(3) The following example is given in [HRW5, Section 4]. Let $n = s = m = 2$, let $f_1 = (y_1 + \tau_1)^2$ and $f_2 = (y_1 + \tau_1)(y_2 + \tau_2)$. It is shown in [HRW5] for this example that $B \subsetneq A$ and that both A and B are non-Noetherian.

The following lemma follows from [P, Proposition 2.1] in the case of one indeterminate x, so in the case where $T = R[x]$.

3.7 Lemma. *Let R be a Noetherian ring, let $x_1, \ldots, x_n$ be indeterminates over R, and let $T = R[x_1, \ldots x_n]$. Suppose $f \in T - R$ is such that the constant term of f is zero. Then the following are equivalent:*

(1) *$R[f] \to T$ is flat.*

(2) *$R[f] \to T$ is faithfully flat.*

(3) *For each maximal ideal q of R, we have $qT \cap R[f] = qR[f]$.*

(4) *The coefficients of f generate the unit ideal of R.*

Proof. (1) $\implies$ (2): It suffices to show for $P \in \operatorname{Spec}(R[f])$ that $PT \neq T$. Let $q = P \cap R$ and let $k(q)$ denote the fraction field of R/q. Since $R[f] \to T$ is flat, tensoring with $k(q)$ gives injective maps

$$k(q) \to k(q) \otimes_R R[f] \cong k(q)[f'] \xrightarrow{\varphi} k(q) \otimes_R T \cong k(q)[x_1, \dots, x_n],$$

where f' is the image of f in $k(q)[x_1, \dots, x_n]$. The injectivity of φ implies f' has positive total degree as a polynomial in $k(q)[x_1, \dots, x_n]$.

The image p' of P in $k(q)[f']$ is either zero or a maximal ideal of $k(q)[f']$. It suffices to show $p'k(q)[x_1, \dots, x_n] \neq k(q)[x_1, \dots, x_n]$. If $p' = 0$, this is clear. Otherwise p' is generated by a nonconstant polynomial $h(f')$ and $p'k(q)[x_1, \dots, x_n]$ is generated by $h(f'(x_1, \dots, x_n))$ which has total degree equal to $\deg(h)\deg(f') > 0$. Thus (1) implies (2).

(2) $\implies$ (3): This follows from Theorem 7.5 (ii) of [M2].

(3) $\implies$ (4) : If the coefficients of f were contained in a maximal ideal q of R, then $f \in qT \cap R[f]$, but $f \notin qR[f]$.

(4) $\implies$ (1): Let v be another indeterminate and consider the commutative diagram

$$\begin{tikzcd} R[v] \arrow[r] \arrow[d, "\pi"'] & T[v] = R[x_1, \dots, x_n, v] \arrow[d, "\pi'"'] \\ R[f] \arrow[r, "\varphi"] & \frac{R[x_1, \dots, x_n, v]}{(v - f(x_1, \dots, x_n))}. \end{tikzcd}$$

where π maps $v \to f$ and π' is the canonical quotient homomorphism. By [M1, Corollary 2, p. 152], φ is flat if the coefficients of $f - v$ generate the unit ideal of $R[v]$. Moreover, the coefficients of $f - v$ as a polynomial in $x_1, \dots, x_n$ with coefficients in $R[v]$ generate the unit ideal of $R[v]$ if and only if the nonconstant coefficients of f generate the unit ideal of R. $\square$

We observe in Proposition 3.8 that one direction of (3.7) holds for more than one polynomial: see also [P, Theorem 3.8] for a related result concerning flatness.

3.8 Proposition. *Assume the notation of (3.7) except that $f_1, \dots, f_m \in T$ are polynomials in $x_1, \dots, x_n$ with coefficients in R and $m \geq 1$. If the inclusion map $\varphi : S = R[f_1, \dots f_m] \to T$ is flat, then the nonconstant coefficients of each of the f_i generate the unit ideal of R.*

Proof. Since $f_1, \dots, f_m$ are algebraically independent over $\mathcal{Q}(R) = K$, for every $1 \leq i \leq m$, the inclusion $R[f_i] \hookrightarrow R[f_1, \dots, f_m]$ is flat. If $S \longrightarrow T$ is flat, so is the composition $R[f_i] \longrightarrow S = R[f_1, \ldots, f_m] \longrightarrow T$ and the statement follows from Proposition 3.7. $\square$

3.9 Theorem. *Assume the notation of (3.1). If $m = 1$, that is, there is only one polynomial $f_1 = f$, then*

(1) *The map $S \to T_a$ is flat $\iff$ the nonconstant coefficients of f generate the unit ideal in R_a,*

(2) *Either of the conditions in (1) implies the constructed ring A is Noetherian and $A = B$.*

(3) *B is Noetherian and $A = B$ $\iff$ for every prime ideal Q^* in R^* with $a \notin Q^*$, the nonconstant coefficients of f generate the unit ideal in R_q, where $q := Q^* \cap R$.*

(4) *If the nonconstant coefficients of $f_1 = f$ generate an ideal L of R_a of height d, then the map $S \to R_a^*$ satisfies LF_{d-1}, but not LF_d.*

Proof. Item (1) follows from Lemma 3.7 for the ring R_a with $x_i = \tau_i$.

By Theorems 2.2 and 3.2, the first condition in item (1) implies item (2).

For item (3), suppose the nonconstant coefficients of f generate the unit ideal of R_q. Then by Lemma 3.7, $R_q[f] \to R_q[\tau_1, \ldots, \tau_n]$ is flat. Since $R_q[\tau_1, \ldots, \tau_n] \to R^*_{Q^*}$ is flat, $R_q[f] \to R^*_{Q^*}$ is also flat. For the other direction, suppose there exists $Q^* \in \operatorname{Spec} R^*$ with $a \notin Q^*$ such that the nonconstant coefficients of f are in qR_q, where $q = Q^* \cap R$. If $R[f] \to R^*_{Q^*}$ were flat, then, since $qR^*_{Q^*} \neq R^*_{Q^*}$, we would have $qR^*_{Q^*} \cap R[f] = qR[f]$. This would imply $f \in qR^*_{Q^*} \cap R[f]$, but $f \notin qR[f]$, a contradiction.

For item (4), if $Q^* \in \operatorname{Spec}(R_a^*)$ the map $S \to (R_a^*)_{Q^*}$ is not flat if and only if $L \subseteq Q^*$. By hypothesis there exists such a prime ideal of height d, but no such prime ideal of height less than d.

3.10 Example. With the notation of (3.5), let $m = 1$ and assume that n and s are each greater than or equal to d. Then $f_1 = f := y_1\tau_1 + \cdots + y_d\tau_d$ gives an example where $S \to T_x$ satisfies LF_{d-1}, but fails to satisfy LF_d. For $d \geq 2$ this gives examples where $A = B$, i.e., A is "limit-intersecting", but is not Noetherian.

The following is a related even simpler example: In the notation of (3.5), let $m = 1, n = 1$, and $s = 2$; that is, $R = k[x, y_1, y_2]_{(x,y_1,y_2)}$ and $\tau \in xk[[x]]$. If $f_1 = f = y_1\tau + y_2\tau^2$, then the constructed intersection domain $A := R^* \cap k(x, y_1, y_2, f)$

is not Noetherian. Thus we have a situation where $B = A$ is not Noetherian. This gives a simpler example of such behavior than the example given in Section 4 of [HRW2].

In dimension two (the two variable case), Valabrega proved the following.

3.11 Proposition [V, Prop. 3]**.** *For $R = k[x,y]_{(x,y)}$ with completion $\widehat{R} = k[[x,y]]$, if L is a field between the fraction field of R and the fraction field F of $k[y]\,[[x]]$, then $A = L \cap \widehat{R}$ is a two-dimensional regular local domain with completion $\widehat{R}$.*

Example 3.10 shows that the dimension three analog to Valabrega's result fails. With $R = k[x,y_1,y_2]_{(x,y_1,y_2)}$ the field $L = k(x,y_1,y_2,f)$ is between $k(x,y_1,y_2)$ and the fraction field of $k[y_1,y_2]\,[[x]]$, but $L \cap \widehat{R} = L \cap R^*$ is not Noetherian.

3.12 Remark. With the notation of (3.1), it can happen that $\varphi_a : S \to T_a$ is not flat, but $\alpha : S \to R_a^*$ is flat. For example, using the notation of (3.5), let $R := k[x,y]$, where k is a field and x,y are indeterminates over k. Let $\sigma,\tau \in xk[[x]]$ be such that x,σ,τ are algebraically independent over k, let $T := R[\sigma,\tau]$, and let $S := R[\sigma,\sigma\tau]$. Then $\varphi_x : S \to T_x$ is not flat since σT_x is a height-one prime such that $\sigma T_x \cap S = (\sigma,\sigma\tau)S$ has height 2. To see that R_x^* is flat over S, observe that $\dim R_x^* = 1$ and if $Q^* \in \operatorname{Spec} R_x^*$, then $Q^* \cap k[x,\sigma,\sigma\tau] = (0)$. Therefore $\operatorname{ht}(Q^* \cap S) \leq 1$.

4. Flatness of maps of polynomial rings.

4.1 Proposition. *Let k be a field, let $x_1,\dots,x_n$ be indeterminates over k, and let $f_1,\dots,f_m \in k[x_1,\dots,x_n]$ be algebraically independent over k. Consider the embedding $\varphi : S := k[f_1,\dots,f_m] \hookrightarrow T := k[x_1,\dots,x_n]$ and let J denote the Jacobian ideal of φ. Then*

(1) $\mathcal{F}_{\min} \subseteq \{Q \in \operatorname{Spec} T : J \subseteq Q,\ \operatorname{ht} Q \leq m-1$ *and* $\operatorname{ht} Q < \operatorname{ht}(Q \cap S)\}$.

(2) *φ is flat $\iff$ for every $Q \in \operatorname{Spec}(T)$ such that* $\operatorname{ht}(Q) \leq m-1$ *and* $J \subseteq Q$ *we have* $\operatorname{ht}(Q \cap S) \leq \operatorname{ht}(Q)$.

(3) *If* $\operatorname{ht} J \geq m$, *then φ is flat.*

Proof. For item 1, if $\operatorname{ht}(Q) \geq m$, then $\operatorname{ht}(Q \cap S) \leq \dim(S) = m \leq \operatorname{ht}(Q)$, so by (2.3) $S \to T_Q$ is flat. Therefore $Q \notin \mathcal{F}_{\min}$. Item 1 now follows from (2.7.3).

The ($\Longrightarrow$) direction of item 2 is clear [M2, Theorem 9.5]. For ($\Longleftarrow$) of item 2 and for item 3, it suffices to show $\mathcal{F}_{\min}$ is empty and this holds by item 1. $\square$

The following is an immediate corollary to (4.1).

4.2 Corollary. *Let k be a field, let $x_1, \dots, x_n$ be indeterminates over k and let $f, g \in k[x_1, \dots, x_n]$ be algebraically independent over k. Consider the embedding $\varphi : S := k[f, g] \hookrightarrow T := k[x_1, \dots, x_n]$ and let J be the associated Jacobian ideal. Then*

(1) $\mathcal{F}_{\min} \subseteq \{$*minimal primes Q of J with* $\operatorname{ht}(Q \cap S) > \operatorname{ht} Q = 1$ $\}$.

(2) φ *is flat* $\iff$ *for every height-one prime ideal $Q \in \operatorname{Spec} T$ such that $J \subseteq Q$ we have* $\operatorname{ht}(Q \cap S) \le 1$.

(3) *If* $\operatorname{ht}(J) \ge 2$, *then φ is flat.*

In the case where k is algebraically closed, another argument can be used for (4.2.2): Each height-one prime ideal $Q \in \operatorname{Spec} T$ has the form $Q = hT$ for some element $h \in T$. If $\operatorname{ht}(P \cap S) = 2$, then $Q \cap S$ has the form $(f - a, g - b)S$, where $a, b \in k$. Thus $f - a = f_1 h$ and $g - b = g_1 h$ for some $f_1, g_1 \in T$. Now the Jacobian ideal of f, g is the same as the Jacobian ideal of $f - a, g - b$ and an easy computation shows this has h as a factor. Thus Q contains the Jacobian ideal, and so by assumption, $\operatorname{ht}(Q \cap S) \le 1$, a contradiction.

4.3 Examples. Let k be a field of characteristic different from 2 and let x, y, z be indeterminates over k.

(1) With $f = x$ and $g = xy^2 - y$, consider $S := k[f, g] \xrightarrow{\varphi} T := k[x, y]$. Then $J = (2xy - 1)T$. Since $\operatorname{ht}((2xy - 1)T \cap S) = 1$, φ is flat. Hence $J \subsetneq F = T$.

(2) With $f = x$ and $g = yz$, consider $S := k[f, g] \xrightarrow{\varphi} T := k[x, y, z]$. Then $J = (y, z)T$. Since $\operatorname{ht} J \ge 2$, φ is flat. Again $J \subsetneq F = T$.

We are interested in extending Prop. 4.1 to the case of polynomial rings over a Noetherian domain. In this connection we first consider behavior with respect to prime ideals of R in a situation where the extension (1.2) is flat.

4.4 Proposition. *Let R be a commutative ring, let $x_1, \dots, x_n$ be indeterminates over R, and let $f_1, \dots, f_m \in R[x_1, \dots, x_n]$ be algebraically independent over R. Consider the embedding $\varphi : S := R[f_1, \dots, f_m] \hookrightarrow T := R[x_1, \dots, x_n]$.*

(1) *If $\mathbf{p} \in \operatorname{Spec} R$ and $\varphi_{\mathbf{p}T} : S \to T_{\mathbf{p}T}$ is flat, then $\mathbf{p}S = \mathbf{p}T \cap S$ and the images $\overline{f_i}$ of the f_i in $T/\mathbf{p}T \cong (R/\mathbf{p})[x_1, \dots, x_n]$ are algebraically independent over $R/\mathbf{p}$.*

(2) *If φ is flat, then for each* $\mathbf{p} \in \operatorname{Spec}(R)$ *we have* $\mathbf{p}S = \mathbf{p}T \cap S$ *and the images* $\overline{f_i}$ *of the* f_i *in* $T/\mathbf{p}T \cong (R/\mathbf{p})[x_1, \dots, x_n]$ *are algebraically independent over* $R/\mathbf{p}$.

Proof. Item 2 follows from item 1, so it suffices to prove item 1. Assume that $T_{\mathbf{p}T}$ is flat over S. Then $\mathbf{p}T \neq T$ and it follows from [M2, Theorem 9.5] that $\mathbf{p}T \cap S = \mathbf{p}S$. If the $\overline{f_i}$ were algebraically dependent over $R/\mathbf{p}$, then there exist indeterminates $t_1, \dots, t_m$ and a polynomial $G \in R[t_1, \dots, t_m] - \mathbf{p}R[t_1, \dots, t_m]$ such that $G(\overline{f_1}, \dots, \overline{f_m}) \in \mathbf{p}T$. This implies $G(f_1, \dots, f_m) \in \mathbf{p}T \cap S$. But $f_1, \dots, f_m$ are algebraically independent over R and $G(t_1, \dots, t_m) \notin \mathbf{p}R[t_1, \dots, t_m]$ implies $G(f_1, \dots, f_m) \notin \mathbf{p}S = \mathbf{p}T \cap S$, a contradiction. □

4.5 Proposition. *Let R be a Noetherian integral domain, let $x_1, \dots, x_n$ be indeterminates over R and let $f_1, \dots, f_m \in R[x_1, \dots, x_n]$ be algebraically independent over R. Consider the embedding $\varphi : S := R[f_1, \dots, f_m] \hookrightarrow T := R[x_1, \dots, x_n]$ and let J denote the Jacobian ideal of φ. Then*

(1) $\mathcal{F}_{\min} \subseteq \{Q \in \operatorname{Spec} T : J \subseteq Q, \dim(T/Q) \geq 1$ *and* $\operatorname{ht}(Q \cap S) > \operatorname{ht} Q\}$.

(2) φ *is flat* $\iff$ $\operatorname{ht}(Q \cap S) \leq \operatorname{ht}(Q)$ *for every nonmaximal* $Q \in \operatorname{Spec}(T)$ *with* $J \subseteq Q$.

(3) *If* $\dim R = d$ *and* $\operatorname{ht} J \geq d + m$, *then* φ *is flat.*

Proof. For item 1, suppose $Q \in \mathcal{F}_{\min}$ is a maximal ideal of T. Then $\operatorname{ht} Q < \operatorname{ht}(Q \cap S)$ by (2.4.2). By localizing at $R - (R \cap Q)$, we may assume that R is local with maximal ideal $Q \cap R := \mathbf{m}$. Since Q is maximal, T/Q is a field finitely generated over $R/\mathbf{m}$. By the Hilbert Nullstellensatz [M2, Theorem 5.3], T/Q is algebraic over $R/\mathbf{m}$ and $\operatorname{ht}(Q) = \operatorname{ht}(\mathbf{m}) + n$. It follows that $Q \cap S = P$ is maximal in S and $\operatorname{ht}(P) = \operatorname{ht}(\mathbf{m}) + m$. But the algebraic independence hypothesis for the f_i implies $m \leq n$. This is a contradiction. Therefore item 1 follows from (2.7.3).

The ($\implies$) direction of item 2 is clear. For ($\impliedby$) of item 2 and for item 3, it suffices to show the set $\mathcal{F}_{\min}$ is empty, and this follows from item 1. □

As an immediate corollary to (2.7) and (4.5), we have:

4.6 Corollary. *Let R be a Noetherian integral domain, let $x_1, \dots, x_n$ be indeterminates over R and let $f_1, \dots, f_m \in R[x_1, \dots, x_n]$ be algebraically independent over R. Consider the embedding $\varphi : S := R[f_1, \dots, f_m] \hookrightarrow T := R[x_1, \dots, x_n]$ and let J be* *the associated Jacobian ideal. Then φ is flat if for every nonmaximal $Q \in \operatorname{Spec}(T)$ such that $J \subseteq Q$ we have $\operatorname{ht}(Q \cap S) \leq \operatorname{ht}(Q)$.*

Also as a corollary of (2.7) and (4.5) we have:

4.7 Corollary. *Let R be a Noetherian ring, let $x_1, \dots, x_n$ be indeterminates over R and let $f_1, \dots, f_m \in R[x_1, \dots, x_n]$ be algebraically independent over R. Consider the embedding $\varphi : S := R[f_1, \dots, f_m] \hookrightarrow T := R[x_1, \dots, x_n]$, let J be the Jacobian ideal of φ and let F be the (reduced) ideal which describes the nonflat locus of φ as in (2.4.2). Then $J \subseteq F$ and either $F = T$, that is, φ is flat, or dim$(T/Q) \geq 1$, for all $Q \in \operatorname{Spec}(T)$ which are minimal over F.*

4.8 Proposition. *Let R be a Noetherian integral domain containing a field of characteristic zero. Let $x_1, \dots, x_n$ be indeterminates over R and let $f_1, \dots, f_m \in R[x_1, \dots, x_n]$ be algebraically independent over R. Consider the embedding $\varphi : S := R[f_1, \dots, f_m] \hookrightarrow T := R[x_1, \dots, x_n]$ and let J be the associated Jacobian ideal. Then*

(1) *If $\mathbf{p} \in \operatorname{Spec} R$ and $J \subseteq \mathbf{p}T$, then $\mathbf{p}T \in \mathcal{F}$, i.e., $\varphi_{\mathbf{p}T} S \to T_{\mathbf{p}T}$ is not flat.*
(2) *If the embedding $\varphi : S \hookrightarrow T$ is flat, then for every $\mathbf{p} \in \operatorname{Spec}(R)$ we have $J \nsubseteq \mathbf{p}T$.*

Proof. Item 2 follows from item 1, so it suffices to prove item 1. Let $\mathbf{p} \in \operatorname{Spec} R$ with $J \subseteq \mathbf{p}T$, and suppose $\varphi_{\mathbf{p}T}$ is flat. Let $\overline{f_i}$ denote the image of f_i in $T/\mathbf{p}T$. Consider

$$\overline{\varphi} : \overline{S} := (R/\mathbf{p})[\overline{f_1}, \dots, \overline{f_m}] \to \overline{T} := (R/\mathbf{p})[x_1, \dots, x_n].$$

By Proposition 4.4, $\overline{f_1}, \dots, \overline{f_m}$ are algebraically independent over $\overline{R} := R/\mathbf{p}$. Since the Jacobian ideal commutes with homomorphic images, the Jacobian ideal of $\overline{\varphi}$ is zero. Thus for each $Q \in \operatorname{Spec} \overline{T}$ the map $\overline{\varphi}_Q : \overline{S} \to \overline{T_Q}$ is not smooth. But taking $Q = (0)$ gives $\overline{T_Q}$ which is a field separable over the fraction field of $\overline{S}$ and hence $\overline{\varphi_Q}$ is a smooth map. This contradiction completes the proof. $\square$

5. Examples.

5.1 Examples. For each positive integer n, we present an example of a 3-dimensional quasilocal unique factorization domain B such that

(1) B is not catenary,

(2) the maximal ideal of B is 2-generated,
(3) B has precisely n prime ideals of height two,
(4) Each prime ideal of B of height two is not finitely generated,
(5) For every non-maximal prime P of B the ring B_P is Noetherian.

The notation for this construction is a localized version of the notation of Section 3.5, with $s = 1$. Thus k is a field, $R = k[x, y]_{(x,y)}$ is a 2-dimensional regular local ring and $R^* = k[y]_{(y)}[[x]]$ is the (x)-adic completion of R. Let $\tau = \sum_{j=1}^{\infty} c_j x^j \in xk[[x]]$ be algebraically independent over $k(x)$. Let $p_i \in R - xR$ be such that $p_i R^*$ are n distinct prime ideals. For example, we could take $p_i = y - x^i$. Let $q = p_1 \cdots p_n$. We set $f := q\tau$ and consider the injective R-algebra homomorphism $S = R[f] \hookrightarrow R[\tau] = T$.

Let B be the nested union domain associated to f as in (2.1). If $\tau_r = \sum_{j=r+1}^{\infty} \frac{c_j x^j}{x^r}$ is the r^{th} endpiece of τ, then $\rho_r := q\tau_r$ is the r^{th} endpiece of f. For each $r \in \mathbb{N}$, let $B_r = R[\rho_r]_{(x,y,\rho_r)}$. Then each B_r is a 3-dimensional regular local ring and $B = \bigcup_{r=1}^{\infty} B_r$.

The map $\alpha : S \to R_x^*$ is not flat since $p_i R_x^*$ is a height-one prime and $p_i R_x^* \cap S = (p_i, f)S$ is of height two. By Theorem 2.2, B is not Noetherian. By [HRW4, Theorem 4.5], B is a quasilocal unique factorization domain. Moreover, by [HRW4, Theorem 4.4], for each $t \in \mathbb{N}, x^t B = x^t R^* \cap B$ and $R/x^t R = B/x^t B = R^*/x^t R^*$. It follows that the maximal ideal of B is $(x, y)B$. If $P \in \operatorname{Spec} B$ is such that $P \cap R = (0)$, then because the field of fractions $K(f)$ of B has transcendence degree one over the field of fractions K of R, $\operatorname{ht}(P) \leq 1$ and hence because B is a UFD, P is principal.

Claim 1. *Let I be an ideal of B and let $t \in \mathbb{N}$. If $x^t \in IR^*$, then $x^t \in I$.*

Proof. There exist elements $b_1, \dots, b_s \in I$ such that $IR^* = (b_1, \dots, b_s)R^*$. If $x^t \in IR^*$, there exist $\alpha_i \in R^*$ such that

$$x^t = \alpha_1 b_1 + \cdots + \alpha_s b_s.$$

We have $\alpha_i = a_i + x^{t+1}\lambda_i$, where $a_i \in B$ and $\lambda_i \in R^*$. Thus

$$x^t[1 - x(b_1\lambda_1 + \cdots + b_s\lambda_s)] = a_1 b_1 + \cdots + a_s b_s \in B.$$

Since $x^t R^* \cap B = x^t B$, $\gamma := 1 - x(b_1\lambda_1 + \cdots + b_s\lambda_s) \in B$. Moreover, γ is invertible in R^* and hence also in B. It follows that $x^t \in I$. $\square$

To examine more closely the prime ideal structure of B, it is useful to consider the inclusion map $B \hookrightarrow A := R^* \cap K(f)$ and the map $\operatorname{Spec} A \to \operatorname{Spec} B$.

5.2 Proposition. *With the notation of Example 5.1 and $A = R^* \cap K(f)$, we have*

(1) *A is a two-dimensional regular local domain with maximal ideal $\mathbf{m}_A = (x, y)A$.*

(2) *$\mathbf{m}_A$ is the unique prime of A lying over $\mathbf{m}_B = (x, y)B$, the maximal ideal of B.*

(3) *If $P \in \operatorname{Spec} B$ is nonmaximal, then $\operatorname{ht}(PR^*) \leq 1$ and $\operatorname{ht}(PA) \leq 1$. Thus every nonmaximal prime of B is contained in a nonmaximal prime of A.*

(4) *If $P \in \operatorname{Spec} B$ and $xq \notin P$, then $\operatorname{ht} P \leq 1$.*

(5) *If $P \in \operatorname{Spec} B$, $\operatorname{ht} P = 1$ and $P \cap R \neq 0$, then $P = (P \cap R)B$.*

Proof. By Proposition 3.11 (the result of Valabrega) $A := R^* \cap K(f)$ is a two-dimensional regular local domain having the same completion as R and R^*. This proves item 1. Since $B/xB = A/xA = R^*/xR^*$, $\mathbf{m}_A = (x, y)A$ is the unique prime of A lying over $\mathbf{m}_B = (x, y)B$. Thus item 2 holds and also item 3 if $x \in P$. To see (3), it remains to consider $P \in \operatorname{Spec} B$ with $x \notin P$. By Claim 1, for all $t \in \mathbb{N}$, $x^t \notin PR^*$. Thus $\operatorname{ht}(PR^*) \leq 1$. Since $A \hookrightarrow R^*$ is faithfully flat, $\operatorname{ht}(PA) \leq 1$.

For (4), we see by (3) that $\operatorname{ht}(PA) \leq 1$. Let $Q \in \operatorname{Spec} A$ be a height-one prime ideal such that $P \subseteq Q$. Since $xq \notin P$, we have $B_P = S_{P \cap S} = T_{Q \cap T} = A_Q$, where $S = R[f]$ and $T = R[\tau]$. Thus $\operatorname{ht}(P) \leq 1$. For (5), if $x \in P$, then $P = xB$ and the statement is clear. Assume $x \notin P$. Since B_x is a localization of $(B_r)_x$, we have $(P \cap R)B_r = P \cap B_r$ for all $r \in \mathbb{N}$. Thus $P = (P \cap R)B$. □

We observe that the DVRs B_{xB} and A_{xA} are equal. Moreover, A is the nested union $\bigcup_{r=1}^{\infty} R[\tau_n]_{(x,y,\tau_n)}$ of 3-dimensional regular local domains. Since A is a two-dimension regular local domain each nonmaximal prime of A is principal. If pA is a height-one prime of A with $pA \notin \{p_1A, \dots p_nA\}$, then $A_{pA} = B_{pA \cap B}$ and $\operatorname{ht}(pA \cap B) = 1$. We observe in Claim 2 that $p_iA \cap B$ has height two and is not finitely generated.

Claim 2. *Let p_i be one of the prime factors of q. Then p_iB is prime in B. Moreover*

(1) *p_iB and $Q_i := (p_i, \rho_1, \rho_2, \dots)B = p_iA \cap B$ are the only primes of B lying*

over p_iR in R,

(2) *Q_i is of height two and is not finitely generated.*

Proof. We use that $B = \bigcup_{r=1}^{\infty} B_r$, where $B_r = R[\rho_r]_{(x,y,\rho_r)}$ is a 3-dimensional regular local ring. For each $r \in \mathbb{N}$, p_iB_r is prime in B_r. Hence p_iB is a height-one prime ideal of B, for $i = 1, \dots, n$. Since $\rho_r = q\tau_r$, $p_iA \cap B_r = (p_i, \rho_r)B_r$ is a height-two prime ideal of the 3-dimensional regular local domain B_r. Therefore $Q_i := (p_i, \rho_1, \rho_2, \dots)B = p_iA \cap B$ is a nested union of prime ideals of height two, so $\mathrm{ht}(Q_i) \leq 2$. Since p_iB is a nonzero prime ideal properly contained in Q_i, $\mathrm{ht}(Q_i) = 2$. Moreover $x \notin (p_i, \rho_r)B_r$ for each r, so $x \notin Q_i$. Hence for each $r \in \mathbb{N}$, $\rho_{r+1} \notin (p_i, \rho_r)B$ and Q_i is not finitely generated. $\square$

Since $x \notin Q_i$ and $B[1/x]$ is a localization of the Noetherian domain $B_n[1/x]$, we see that B_{Q_i} is Noetherian. Since the Q_i are the only prime ideals of B of height two and B is a UFD, B_P is Noetherian for every non-maximal prime P.

This completes the presentation of Examples 5.1. With regard to the birational inclusion $B \hookrightarrow A$ and the map $\operatorname{Spec} A \to \operatorname{Spec} B$, we remark that the following holds: Each Q_i contains infinitely many height-one primes of B that are the contraction of primes of A and infinitely many that are not. Among the primes that are not contracted from A are the p_iB. In the terminology of [ZS, page 325], P is *not lost* in A if $PA \cap B = P$. Since $p_iA \cap B = Q_i$ properly contains p_iB, p_iB is lost in A. Since $(x, y)B$ is the maximal ideal of B and $(x, y)A$ is the maximal ideal of A and B is integrally closed, a version of Zariski's Main Theorem [Pe], [Ev], implies that A is not essentially finitely generated as a B-algebra.

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