THE FIRST MAYR-MEYER IDEAL

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Summary. This paper gives a complete primary decomposition of the first, that is, the smallest, Mayr-Meyer ideal, its radical, and the intersection of its minimal components. The particular membership problem which makes the Mayr-Meyer ideals' complexity doubly exponential in the number of variables is here examined also for the radical and the intersection of the minimal components. It is proved that for the first Mayr-Meyer ideal the complexity of this membership problem is the same as for its radical. This problem was motivated by a question of Bayer, Huneke and Stillman.

Grete Hermann proved in [H] that for any ideal I in an n-dimensional polynomial ring over the field of rational numbers, if I is generated by polynomials $f_1, \ldots, f_k$ of degree at most d, then it is possible to write $f = \sum r_i f_i$, where each r_i has degree at most $\deg f + (kd)^{(2^n)}$. Mayr and Meyer in [MM] found ideals $J(n,d)$ for which a doubly exponential bound in n is indeed achieved. Bayer and Stillman [BS] showed that for these same ideals also any minimal generating set of syzygies has elements of degree which is doubly exponential in n. Koh [K] modified the original ideal to obtain homogeneous quadric ideals with doubly exponential degrees of syzygies and ideal membership coefficients.

Bayer, Huneke and Stillman have raised questions about the structure of these Mayr-Meyer ideals: is the doubly exponential behavior due to the number of minimal primes, to the number of associated primes, or to the structure of one of them? This paper, together with [S], is an attempt at answering these questions. More precisely, the Mayr-Meyer ideal $J(n,d)$ is an ideal in a polynomial ring in $10n+2$ variables whose generators have degree at most $d+2$. This paper analyzes the case $n = 1$ and shows that in this base case the embedded components do not play a role.

Theorem 1 of this paper gives a complete primary decomposition of $J(1,d)$, after which the intersection of the minimal components and the radical come as easy corollaries. The last proposition shows that the complexity of the particular membership problem from

[MM, BS, K] for the radical of $J(1,d)$ is the same as the complexity of the membership problem for $J(1,d)$. Thus at least for the case $n=1$, neither the embedded components nor the non-reducedness play a role in the complexity.

In a developing paper “Primary decomposition of the Mayr-Meyer ideal” [S], partial primary decompositions are determined for the Mayr-Meyer ideals $J(n,d)$ for all $n \geq 2$, $d \geq 1$. Under the assumption that the characteristic of the field does not divide d, for $n \geq 2$, the number of minimal primes is exactly nd^2+20, and the number of embedded primes likewise depends on n and d. However, a precise number of embedded components is not known. The case $n=1$ is very different from the case $n \geq 2$. For example, under the same assumption on the characteristic of the field, the number of minimal primes of the first Mayr-Meyer ideal is $d+4$, and there is exactly one embedded prime. For understanding the asymptotic behavior of the Mayr-Meyer ideals $J(n,d)$, the case $n=1$ may not seem interesting, however, it is a basis of the induction arguments for the behavior of the other $J(n,d)$. Furthermore, the case $n=1$ is computationally and notationally more accessible.

All results of this paper were verified for specific low values of d on Macaulay2.

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The first Mayr-Meyer ideal $J(1,d)$ is defined as follows. Let K be a field, and d a positive integer. In case the characteristic of K is a positive prime p, write $d=d'i$, where i is a power of p, and d' and p are relatively prime integers. In case the characteristic of K is zero, let $d'=d$, $i=1$. For notational simplicity we assume that K contains all the $\frac{d}{i}$th roots of unity. Let $s, f, s_1, f_1, c_1, \ldots, c_4, b_1, \ldots, b_4$ be indeterminates over K, and $R = K[s, f, s_1, f_1, c_1, \ldots, c_4, b_1, \ldots, b_4]$. Note that R has dimension 12. The Mayr-Meyer ideal for $n=1$ is the ideal in R with the generators as follows:

$$\begin{aligned} J = J(1,d) = {} & (s_1 - sc_1, f_1 - sc_4) + \left(c_i\left(s - fb_i^d\right) | i = 1,2,3,4\right) \\ & + \left(fc_1 - sc_2, fc_4 - sc_3, s\left(c_3 - c_2\right), f\left(c_2 b_1 - c_3 b_4\right), fc_2\left(b_2 - b_3\right)\right). \end{aligned}$$

THEOREM 1: *A minimal primary decomposition of $J = J(1,d)$ is as follows:*

$J = (s_1 - sc_1, f_1 - sc_4, c_1, c_2, c_3, c_4)$

$$
\begin{aligned}
&\bigcap_{\alpha} \left(s_1 - sc_1, f_1 - sc_4, c_4 - c_1, c_3 - c_2, c_1 - c_2 b_1^d, s - f b_1^d, b_1 - b_4, b_2 - b_3, b_1^i - \alpha b_2^i\right) \\
&\cap (s_1 - sc_1, f_1 - sc_4, s, f) \\
&\cap \left(s_1 - sc_1, f_1 - sc_4, s, c_1, c_2, c_4, b_3^d, b_4\right) \\
&\cap \left(s_1 - sc_1, f_1 - sc_4, s, c_1, c_4, b_3^d, b_2 - b_3, c_2 b_1 - c_3 b_4\right) \\
&\cap \left(s_1 - sc_1, f_1 - sc_4, s^2, f^2, c_4(s - f b_4^d), c_3(s - f b_3^d), sc_3 - f c_4, c_3^2, c_4^2,\right. \\
&\qquad\qquad \left. c_1 - c_4, c_2 - c_3, b_2 - b_3, b_1 - b_4\right),
\end{aligned}
$$

where the α vary over the $\frac{d}{i}$th roots of unity in K.

It is easy to verify that $J = J(1, d)$ is contained in the intersection, and that all but the last ideal on the right-hand side of the equality are primary. The following lemma proves that the last ideal is primary as well:

LEMMA 2: *The last ideal in the intersection in Theorem 1 is primary.*

Proof: Here is a simple fact: let $x_1, \ldots, x_n$ be variables over a ring A, $S = A[x_1, \ldots, x_n]$, and I an ideal in A. Then I is primary (respectively, prime) if and only if for any $f_1, \ldots, f_n \in A$, $IS + (x_1 - f_1, \ldots, x_n - f_n)S$ is a primary (respectively, prime) ideal in S.

By this fact it suffices to prove that the ideal

$$
L = \left(s^2, f^2, c_4(s - f b_4^d), c_3(s - f b_3^d), sc_3 - f c_4, c_3^2, c_4^2\right)
$$

is primary. Note that $\sqrt{L} = (s, f, c_3, c_4)$ is a prime ideal. It suffices to prove that the set of associated primes of L is $\{\sqrt{L}\}$. It is an easy fact that for any $x \in R$,

$$
\operatorname{Ass}\left(\frac{R}{L}\right) \subseteq \operatorname{Ass}\left(\frac{R}{L : x}\right) \cup \operatorname{Ass}\left(\frac{R}{L + (x)}\right).
$$

In particular, when $x = f$, $L + (f) = \left(s^2, f, sc_4, sc_3, c_3^2, c_4^2\right)$ is clearly primary to $\sqrt{L}$. Thus it suffices to prove that $L : f$ is primary to $\sqrt{L}$.

We fix the monomial lexicographic ordering $s > f > c_4 > c_3 > b_4 > b_3$. Clearly $L : f$ contains $(s^2, f, c_4 - c_3 b_3^d, sc_3, c_3^2)$. If $r \in L : f$, then the leading term of r times f is contained in the ideal of leading terms of L, namely in $(s^2, f^2, sc_4, sc_3, c_3^2, c_4^2, f c_4)$, so that the leading term of r lies in $(s^2, f, c_4, sc_3, c_3^2)$. This proves that $L : f = (s^2, f, c_4 - c_3 b_3^d, sc_3, c_3^2)$. This ideal is clearly primary to $\sqrt{L}$, which proves the lemma. ∎

We next prove that the intersection of ideals in Theorem 1 equals $J = J(1, d)$. Note that it suffices to prove the shortened equality:

$$\begin{aligned}
&(c_1, c_2, c_3, c_4) \\
&\quad \bigcap_\alpha \left(c_4 - c_1, c_3 - c_2, c_1 - c_2 b_1^d, s - f b_1^d, b_1 - b_4, b_2 - b_3, b_1^i - \alpha b_2^i\right) \\
&\quad \cap (s, f) \\
&\quad \cap \left(s, c_1, c_2, c_4, b_3^d, b_4\right) \\
&\quad \cap \left(s, c_1, c_4, b_3^d, b_2 - b_3, c_2 b_1 - c_3 b_4\right) \\
&\quad \cap \left(s^2, f^2, c_4(s - f b_4^d), c_3(s - f b_3^d), s c_3 - f c_4, c_3^2, c_4^2, c_1 - c_4, c_2 - c_3, b_2 - b_3, b_1 - b_4\right) \\
&= \left(c_i \left(s - f b_i^d\right), f c_1 - s c_2, f c_4 - s c_3, s\left(c_3 - c_2\right), f\left(c_2 b_1 - c_3 b_4\right), f c_2\left(b_2 - b_3\right)\right).
\end{aligned}$$

The intersection of the first two rows equals:

$$\begin{aligned}
&(c_1, c_2, c_3, c_4) \cap \left(c_4 - c_1, c_3 - c_2, c_1 - c_2 b_1^d, s - f b_1^d, b_1 - b_4, b_2 - b_3, b_1^d - b_2^d\right) \\
&\quad = \left(c_4 - c_1, c_3 - c_2, c_1 - c_2 b_1^d\right) + (c_1, c_2, c_3, c_4) \cdot \left(s - f b_1^d, b_1 - b_4, b_2 - b_3, b_1^d - b_2^d\right) \\
&\quad = J + \left(c_4 - c_1, c_3 - c_2, c_1 - c_2 b_1^d\right) + c_2 \cdot \left(b_1 - b_4, b_2 - b_3, b_1^d - b_2^d\right).
\end{aligned}$$

This intersected with the third row, namely with (s, f), equals

$$J + (s, f)\left(c_4 - c_1, c_3 - c_2, c_1 - c_2 b_1^d\right) + c_2 (s, f)\left(b_1 - b_4, b_2 - b_3, b_1^d - b_2^d\right).$$

Modulo J,

$$\begin{aligned}
s c_1 &\equiv f c_1 b_1^d \equiv s c_2 b_1^d \equiv f c_2 b_1^d b_2^d \equiv f c_2 b_1^d b_3^d \\
&\equiv f c_3 b_1^{d-1} b_4 b_3^d \equiv s c_3 b_1^{d-1} b_4 \equiv s c_2 b_1^{d-1} b_4 \equiv f c_2 b_1^{d-1} b_4 b_2^d \equiv f c_2 b_1^{d-1} b_4 b_3^d \\
&\equiv f c_3 b_1^{d-2} b_4^2 b_3^d \equiv \cdots \equiv f c_3 b_1^0 b_4^d b_3^d \equiv s c_3 b_4^d \equiv f c_4 b_4^d \equiv s c_4,
\end{aligned}$$

so that $s(c_1 - c_4) \in J$. Also it is clear that $f(c_1 - c_4), s(c_3 - c_2), s(c_1 - c_2 b_1^d) \in J$, that $s c_2 \in J + (f c_2)$, and that $f c_2 (b_2 - b_3) \in J$. Thus the intersection of the ideals in the first three rows of Theorem 1 simplifies to $J + f\left(c_3 - c_2, c_1 - c_2 b_1^d\right) + f c_2\left(b_1 - b_4, b_1^d - b_2^d\right)$. Furthermore, $f c_2 (b_1 - b_4) \in (f(c_3 - c_2)) + J$ and modulo J, $f(c_1 - c_2 b_1^d) \equiv c_2(s - f b_1^d) \equiv f c_2 (b_2^d - b_1^d)$, so that finally the intersection of the first three rows simplifies to

$$J + \left(f(c_3 - c_2), f c_2 (b_1^d - b_2^d)\right).$$

We intersect this with the (shortened) ideal in the fourth row of Theorem 1, namely with $(s, c_1, c_2, c_4, b_3^d, b_4)$, to get

$$\begin{aligned} & J + (fc_2(b_1^d - b_2^d)) + (f(c_3 - c_2)) \cap (s, c_1, c_2, c_4, b_3^d, b_4) \\ &\quad = J + (fc_2(b_1^d - b_2^d)) + f(c_3 - c_2) \cdot (s, c_1, c_2, c_4, b_3^d, b_4) \\ &\quad = J + (fc_2(b_1^d - b_2^d)) + f(c_3 - c_2) \cdot (c_2, b_3^d, b_4). \end{aligned}$$

As modulo J, $f(c_3-c_2)b_4 \equiv fc_2(b_1-b_4)$, and $f(c_3-c_2)b_3^d \equiv fc_3b_3^d - fc_2b_2^d \equiv sc_3 - sc_2 \equiv 0$, the intersection of the first four rows simplifies to

$$J + fc_2\left(b_1^d - b_2^d, c_3 - c_2, b_1 - b_4\right).$$

Next we intersect this with the ideal in the fifth row (of Theorem 1) namely with $(s, c_1, c_4, b_3^d, b_2 - b_3, c_2b_1 - c_3b_4)$, to get:

$$\begin{aligned} & J + \left(fc_2\left(b_1^d - b_2^d, c_3 - c_2, b_1 - b_4\right)\right) \cap \left(s, c_1, c_4, b_3^d, b_2 - b_3, c_2b_1 - c_3b_4\right) \\ &= J + fc_2\left(\left(b_1^d - b_2^d, c_3 - c_2, b_1 - b_4\right) \cap \left(s, c_1, c_4, b_3^d, b_2 - b_3, c_2b_1 - c_3b_4\right)\right) \\ &= J + fc_2\left((c_2b_1 - c_3b_4) + \left(b_1^d - b_2^d, c_3 - c_2, b_1 - b_4\right) \cap \left(s, c_1, c_4, b_3^d, b_2 - b_3\right)\right) \\ &= J + fc_2\left((c_2b_1 - c_3b_4) + \left(b_1^d - b_2^d, c_3 - c_2, b_1 - b_4\right) \cdot \left(s, c_1, c_4, b_3^d, b_2 - b_3\right)\right) \\ &= J + fc_2\left(b_1^d - b_2^d, c_3 - c_2, b_1 - b_4\right) \cdot \left(s, c_1, c_4, b_3^d\right). \end{aligned}$$

As modulo J,

$$\begin{aligned} sc_2(b_1 - b_4) &\equiv fc_3b_3^d(b_1 - b_4) \equiv fb_3^d(c_3 - c_2)b_1 \equiv (sc_3 - fb_2^dc_2)b_1 \equiv s(c_3 - c_2)b_1 \equiv 0, \\ sfc_2(b_1^d - b_2^d) &\equiv f^2c_1b_1^d - s^2c_2 \equiv sfc_1 - sfc_1 = 0, \\ fc_1 &\equiv fc_4, \end{aligned}$$

the intersection of the ideals in the first five rows simplifies to

$$\begin{aligned} & J + fc_2\left(b_1^d - b_2^d, c_3 - c_2, b_1 - b_4\right) \cdot \left(c_1, b_3^d\right) \\ &\quad = J + sc_2\left(b_1^d - b_2^d, c_3 - c_2, b_1 - b_4\right) \\ &\quad = J + sc_2\left(b_1^d - b_2^d\right). \end{aligned}$$

Finally we intersect this intersection of the ideals in the first five rows in the statement of Theorem 1 with the (shortened) last ideal there, namely with $L = (s^2, f^2, c_4(s - fb_4^d), c_3(s - fb_3^d), sc_3 - fc_4, c_3^2, c_4^2, c_1 - c_4, c_2 - c_3, b_2 - b_3, b_1 - b_4)$, to get:

$$J + sc_2\left(b_1^d - b_2^d\right) \cap L.$$

It is easy to see that $L : sc_2$ contains $\sqrt{L}$. As sc_2 is not in L, then $L : sc_2 = \sqrt{L}$, so that the intersection of all the ideals in Theorem 1 equals

$$\begin{aligned} J+sc_2\left(b_1^d-b_2^d\right)\left(L : sc_2\left(b_1^d-b_2^d\right)\right) &= J+sc_2\left(b_1^d-b_2^d\right)\left(\sqrt{L} : \left(b_1^d-b_2^d\right)\right) \\ &= J+sc_2\left(b_1^d-b_2^d\right)\sqrt{L} \\ &= J+sc_2\left(b_1^d-b_2^d\right)(s,f,c_1,c_2,c_3,c_4,b_2-b_3,b_1-b_4) \\ &= J+sc_2\left(b_1^d-b_2^d\right)(f,c_2)\,. \end{aligned}$$

It has been proved that $sfc_2\left(b_1^d-b_2^d\right) \in J$, and similarly $sc_2^2\left(b_1^d-b_2^d\right) \in J$. This proves that the intersection of all the listed ideals in Theorem 1 does equal J. ∎

In order to finish the proof of Theorem 1, it remains to prove that none of the listed components is redundant. The last component is primary to a non-minimal prime, whereas there are no inclusion relations among the rest of the primes. Thus the first $d'+4$ listed components belong to minimal primes and are not redundant. With this it suffices to prove that J has an embedded prime:

Lemma 3: *When $n=1$, $c_4(s-fb_3^d)$ is in every minimal component but not in J. Thus there exists an embedded component.*

Proof: It has been established that $c_4(s-fb_3^d)$ is in every minimal component. Suppose that $c_4(s-fb_3^d)$ is in J. Then

$$\begin{aligned} c_4(s-fb_3^d) = \sum_{i=1}^4 r_i c_i\left(s-fb_i^d\right) &+ r_5(fc_1-sc_2)+r_6(fc_4-sc_3)+r_7 s\left(c_3-c_2\right) \\ &+ r_8 f\left(c_2b_1-c_3b_4\right)+r_9 fc_2\left(b_2-b_3\right), \end{aligned}$$

for some elements r_i in the ring. By the homogeneity of all elements in the two sets of variables $\{s,f\}$ and $\{c_1,c_2,c_3,c_4\}$, without loss of generality each r_i is an element of $K[b_i|i=1,2,3,4]$. Therefore the coefficients of the fc_i, sc_i yield the following equations:

$$\begin{aligned} sc_4 :&\quad 1 = r_4, \\ fc_4 :&\quad -b_3^d = -r_4b_4^d + r_6, \text{ so } r_6 = b_4^d - b_3^d, \\ fc_3 :&\quad 0 = r_3b_3^d + r_8b_4, \text{ so } r_3 = rb_4, r_8 = -rb_3^d \text{ for some } r \in R, \\ sc_3 :&\quad 0 = b_4^d - b_3^d - r_3 - r_7, \text{ so } r_7 = b_4^d - b_3^d - rb_4, \end{aligned}$$

$$
\begin{aligned}
sc_1: &\quad 0 = r_1, \\
fc_1: &\quad 0 = -r_1 b_1^d + r_5, \text{ so } r_5 = 0, \\
sc_2: &\quad 0 = r_2 - b_4^d + b_3^d + rb_4, \text{ so } r_2 = b_4^d - b_3^d - rb_4, \\
fc_2: &\quad 0 = -r_2 b_2^d - rb_3^d b_1 + r_9(b_2 - b_3).
\end{aligned}
$$

After expanding r_2 in the last equation, $0 = -(b_4^d - b_3^d - rb_4)b_2^d - rb_3^d b_1 + r_9\,(b_2 - b_3)$, so that $b_2^d b_3^d \in (b_1, b_4, b_2 - b_3)$, which is a contradiction. ■

As one embedded component has been established, this proves the Theorem. Thus in the case $n = 1$, the Mayr-Meyer ideal $J(1,d)$ has $d' + 4$ minimal primes and one embedded one, and these associated prime ideals are as follows (α varies over the d' th roots of unity):

associated prime ideal	height
$(s_1 - sc_1, f_1 - sc_4, c_1, c_2, c_3, c_4)$	6
$\big(s_1 - sc_1, f_1 - sc_4, c_4 - c_1, c_3 - c_2, c_1 - c_2 b_1^d, s - fb_1^d, b_1 - b_4, b_2 - b_3, b_1 - \alpha b_2\big)$	9
$(s_1 - sc_1, f_1 - sc_4, s, f)$	4
$(s_1 - sc_1, f_1 - sc_4, s, c_1, c_2, c_4, b_3, b_4)$	8
$(s_1 - sc_1, f_1 - sc_4, s, c_1, c_4, b_2, b_3, c_2 b_1 - c_3 b_4)$	8
$(s_1 - sc_1, f_1 - sc_4, s, f, c_1, c_2, c_3, c_4, b_2 - b_3, b_1 - b_4)$	10

The proof of the Theorem also explicitly computes the intersection of the first five rows of the primary decomposition, so that:

PROPOSITION 4: *The intersection of all the minimal components of $J(1,d)$ equals $J + (sc_2(b_1^d - b_2^d))$.* ■

Furthermore, it is straightforward to compute the radical of $J(1,d)$:

PROPOSITION 5: *The radical of $J(1,d)$ equals $J(1,d') + fb_3(c_3 - c_2, c_2(b_1^{d'} - b_2^{d'}))$.*

Proof: It is straightforward to compute the radical of each component. Note that as in the previous proof it suffices to compute the shortened intersection:

$$(c_1, c_2, c_3, c_4)$$

$$\begin{aligned}
&\bigcap_\alpha \left(c_4 - c_1, c_3 - c_2, c_1 - c_2 b_1^d, s - f b_1^d, b_1 - b_4, b_2 - b_3, b_1 - \alpha b_2\right)\\
&\cap (s, f)\\
&\cap (s, c_1, c_2, c_4, b_3, b_4)\\
&\cap (s, c_1, c_4, b_2, b_3, c_2 b_1 - c_3 b_4)\\
= &\left(c_i \left(s - f b_i^d\right), f c_1 - s c_2, f c_4 - s c_3, s\left(c_3 - c_2\right), f\left(c_2 b_1 - c_3 b_4\right)\right)\\
&\quad + \left(f c_2 \left(b_2 - b_3\right), f b_3 (c_3 - c_2), f b_3 c_2 (b_1^{d'} - b_2^{d'})\right).
\end{aligned}$$

As in the proof of the Theorem, the intersection of the first three rows equals $J(1, d') + \left(f(c_3 - c_2), f c_2 (b_1^{d'} - b_2^{d'})\right)$. Intersection with the ideal in the fourth row, namely with $(s, c_1, c_2, c_4, b_3, b_4)$, equals

$$\begin{aligned}
J(1, d') &+ \left(f c_2 (b_1^{d'} - b_2^{d'})\right) + (f(c_3 - c_2)) \cap (s, c_1, c_2, c_4, b_3, b_4)\\
&= J(1, d') + \left(f c_2 (b_1^{d'} - b_2^{d'})\right) + (f(c_3 - c_2)) \cdot (s, c_1, c_2, c_4, b_3, b_4)\\
&= J(1, d') + \left(f c_2 (b_1^{d'} - b_2^{d'})\right) + (f(c_3 - c_2)) \cdot (c_2, b_3, b_4).
\end{aligned}$$

When this is intersected with the ideal in the fifth row, namely with $(s, c_1, c_4, b_2, b_3, c_2 b_1 - c_3 b_4)$, the resulting radical of $J(1, d)$ equals

$$\begin{aligned}
&J(1, d') + (f b_3 (c_3 - c_2))\\
&\qquad + \left(f c_2 (b_1^{d'} - b_2^{d'}), f c_2 (c_3 - c_2), f b_4 (c_3 - c_2)\right) \cap (s, c_1, c_4, b_2, b_3, c_2 b_1 - c_3 b_4)\\
&= J(1, d') + (f b_3 (c_3 - c_2))\\
&\qquad + f c_2 \left(\left(b_1^{d'} - b_2^{d'}, c_3 - c_2, b_1 - b_4\right) \cap (s, c_1, c_4, b_2, b_3, c_2 b_1 - c_3 b_4)\right)\\
&= J(1, d') + (f b_3 (c_3 - c_2)) + f c_2 \left(\left(b_1^{d'} - b_2^{d'}, c_3 - c_2, b_1 - b_4\right) \cdot (s, c_1, c_4, b_2, b_3)\right),
\end{aligned}$$

and by previous computations this simplifies to

$$\begin{aligned}
J(1, d') &+ (f b_3 (c_3 - c_2)) + f c_2 b_3 \left(b_1^{d'} - b_2^{d'}, c_3 - c_2, b_1 - b_4\right)\\
&= J(1, d') + f b_3 (c_3 - c_2, c_2 (b_1^{d'} - b_2^{d'})). \quad \blacksquare
\end{aligned}$$

Mayr and Meyer [MM] observed that whenever the element $s(c_4 - c_1)$ of $J(1, d)$ is expressed as an R-linear combination of the given generators of $J(1, d)$, at least one of the coefficients has degree at least d. In fact, as the proposition below proves, the degree of at least one of the coefficients is at least $2d-1$, and this lower bound is achieved. (See also the proof of Theorem showing that $sc_4 \equiv sc_1$ modulo $J(1,d)$.) Mayr and Meyer also showed the analogues for $n \geq 1$, with degrees of the coefficients depending on $n-1$ doubly exponentially.

Bayer, Huneke and Stillman questioned how much this doubly exponential growth depends on the existence of embedded primes of $J(n,d)$, or on the structure of the components. The proposition below shows that at least for $n=1$, the facts that $J(1,d)$ has an embedded prime and that the minimal components are not radical, do not seem to be crucial for this property:

PROPOSITION 6: *Let I be any ideal between $J(1,d)$ and its radical. Then whenever the element $s(c_4-c_1)$ is expressed as an R-linear combination of the minimal generators of I which include all the given generators of $J(1,d)$, at least one of the coefficients has degree at least $2d-1$.*

Proof: All the cases can be deduced from the case of I being the radical of $J(1,d)$. To simplify the notation it suffices to replace d by d', so that $I = J(1,d) + fb_3(c_3-c_2, c_2(b_1^d - b_2^d))$. Write

$$
\begin{aligned}
s(c_4-c_1) = &\sum_{i=1}^{4} r_i c_i \left(s - fb_i^d\right) + r_5(fc_1 - sc_2) + r_6(fc_4 - sc_3) + r_7 s\left(c_3 - c_2\right) \\
&+ r_8 f\left(c_2 b_1 - c_3 b_4\right) + r_9 f c_2\left(b_2 - b_3\right) + r_{10} f b_3 (c_3 - c_2) + r_{11} f b_3 c_2 (b_1^d - b_2^d)
\end{aligned}
$$

for some elements r_i in the ring. Note that each of the explicit elements of I is homogeneous in the two sets of variables $\{s,f\}$ and $\{c_1,c_2,c_3,c_4\}$. Thus it suffices to prove that in degrees 1 in each of the two sets of variables, one of the coefficients has degree at least $2d-1$. So without loss of generality each r_i is an element of $K[b_i | i = 1,2,3,4]$. Therefore the coefficients of the sc_i, fc_i yield

$$
\begin{aligned}
sc_4 &: \quad 1 = r_4, \\
fc_4 &: \quad 0 = -r_4 b_4^d + r_6, \text{ so } r_6 = b_4^d, \\
sc_1 &: \quad -1 = r_1, \\
fc_1 &: \quad 0 = -r_1 b_1^d + r_5, \text{ so } r_5 = -b_1^d, \\
sc_2 &: \quad 0 = r_2 + b_1^d - r_7, \text{ so } r_7 = r_2 + b_1^d,
\end{aligned}
$$

$$fc_2: \quad 0 = -r_2 b_2^d + r_8 b_1 + r_9\,(b_2 - b_3) - r_{10} b_3 + r_{11} b_3 (b_1^d - b_2^d),$$

$$sc_3: \quad 0 = r_3 - b_4^d + r_7, \text{ so } \ r_3 = b_4^d - b_1^d - r_2,$$

$$fc_3: \quad 0 = -r_3 b_3^d - r_8 b_4 + r_{10} b_3.$$

The last equation implies that $r_8 = r_8' b_3$, so that the equations for the coefficients of fc_3 and fc_2 can be rewritten as

$$\begin{aligned} r_{10} &= r_3 b_3^{d-1} + r_8' b_4 = (b_4^d - b_1^d) b_3^{d-1} - r_2 b_3^{d-1} + r_8' b_4, \\ 0 &= -r_2 b_2^d + r_8' b_3 b_1 + r_9\,(b_2 - b_3) - (b_4^d - b_1^d) b_3^d + r_2 b_3^d - r_8' b_4 b_3 + r_{11} b_3 (b_1^d - b_2^d), \\ &= -r_2 (b_2^d - b_3^d) + r_8' b_3 (b_1 - b_4) + r_9 (b_2 - b_3) - (b_4^d - b_1^d) b_3^d + r_{11} b_3 (b_1^d - b_2^d). \end{aligned}$$

Thus $r_8' b_3 (b_1 - b_4) \in (b_2 - b_3, b_3^d (b_1^d - b_4^d), b_1^d - b_2^d)$, so that $r_8' \in (b_2 - b_3, b_3^{d-1} \frac{b_1^d - b_4^d}{b_1 - b_4}, b_1^d - b_2^d)$. If $r_8' \in (b_2 - b_3, b_1^d - b_2^d)$. Then $(b_4^d - b_1^d) b_3^d \in (b_2 - b_3, b_1^d - b_2^d)$, which is a contradiction. Thus r_8' has a multiple of $b_3^{d-1} \frac{b_1^d - b_4^d}{b_1 - b_4}$ as a summand, so r_8' has degree at least $2d - 2$, so that r_8 has degree at least $2d - 1$. In fact, by setting all the free variables r_2, r_{11} to zero, the maximum degree of the coefficients r_i is $2d - 1$. ∎

Note that in the proof above it is possible to have both $r_{10} = r_{11} = 0$ and the degrees of the r_i still at most $2d - 1$, with $2d - 1$ attained on some r_i. (Lemma 2.3 of [BS] erroneously claims that the degree of some r_i is at least $2d$.)

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