Why does .999999999 ... = 1?

For some reason it seems easier to accept that $0.33333\cdots = \frac{1}{3}$ than that $0.99999\cdots = 1$, even though the latter is gotten from the former just by multiplying both sides by 3.

The presentation in class about real numbers was intended only to convey a general idea of how to represent and work with real numbers, and did not go into complete detail. If we had, for example, carefully defined how to divide and multiply infinite decimals, then the above"$0.33\cdots = \frac{1}{3}$, and multiply by 3" argument would be completely justified.

Alternatively, once it is carefully defined how to subtract infinite decimals, it turns out that

$$1.00000\cdots - 0.99999\cdots = 0.00000\ldots,$$

so the difference between 1 and $0.99999\ldots$ is indeed 0. See below for some more details about subtraction.

Making precise all the details about performing arithmetic operations on infinite decimals is somewhat tedious, and is best left to a later course, for those who'd like to make the effort to dig down to the foundations of mathematics in logic and set theory. The essential point is to establish the algebraic, order, and completeness properties listed in Appendix 4 of the textbook, because any statement ordinarily made about real numbers can be logically deduced from those few properties.[1] (Compare with the deduction of all kinds of geometric properties from just a few axioms and postulates in Euclid's treatment.) Any two "complete ordered fields," that is, systems of objects having those properties, are essentially the same, just as any two chess sets are essentially the same, even though they may look quite different. That's why we have a number of different constructions, or "models," of the real number system, for example those mentioned in Appendix 4.

Incidentally, I don't like the text's statement that the "real meaning" of infinite decimals has to do with limits. You can build up the infinite decimal model of the real numbers without ever mentioning limits. (Technically, the decimal part of a real number is a function defined on the set of natural numbers, taking values in the set of digits $\{0, 1, 2, \ldots, 9\}$. Or, if you use binary decimals, so that the only digits are 0 and 1, you can identify any decimal in the interval $[0, 1]$ with a subset of the natural numbers, namely the one consisting of those n such that 1 appears in the n-th decimal place, and then get an arbitrary real number by adding an integer to such a decimal.) Later on, there will emerge an interpretation of the decimal representation of a real number $a_0.a_1a_2\ldots a_n\ldots$ as the limit of the sequence of finite decimals $\big(a_0.a_1a_2\ldots a_n\big)_{n=1,2,3,\ldots}$; and though this enhances the understanding of infinite decimals, it is an *interpretation,* not the "real meaning" (whatever that really means).

Now here are a few basic facts about subtraction.

(Terminology.) *Nonnegative* real numbers are infinite decimals which can be represented with a nonnegative integer part, so that, geometrically, they are the ones which either are 0 or lie to the right of 0 on the line. The remaining real numbers are *negative.* Negative real numbers are always written as negatives of positive (=nonnegative and nonzero) real numbers. (For example, $-3.14159\ldots$)

In what follows, to avoid extra detail, we confine our attention to nonnegative real numbers.

First, to decide if one real number is *less than or equal to* another, look at the successive decimal places until you come to one where the two numbers differ, and then see which is less in that place. That one will be *less* than the other *unless* the smaller digit in that place is one less than the larger digit, and the larger digit is followed by 000...(forever) while the smaller digit is followed by 999...(forever), in which case the two numbers are actually *equal.* So, for example, $3.1415900000\ldots$ is equal to $3.1415899999\ldots$.

[1] The least upper bound of a nonempty bounded set of reals can be constructed, one decimal place at a time, by a procedure somewhat like the one in the proof of the Intermediate Value Theorem given in class.

The simplest case of this happening is $1.00000\cdots = 0.99999\ldots$.

Let's show that the difference between these two numbers is 0.

Given two nonnegative infinite decimals $x \geq y$ how do we describe $x - y$? Say $x = a_0.a_1a_2\ldots a_n\ldots$, $y = b_0.b_1b_2\ldots b_n\ldots$. Then *define* $x - y$ to be $c_0.c_1c_2\ldots c_n\ldots$ where for each n, $c_0.c_1c_2\ldots c_n$—terminating at the n-th place—is described as follows. (The idea, roughly, is to do what a calculator does: round off to a bit more than the degree of accuracy you're looking for, then do the subtraction. But to get the exact answer, we have to proceed more carefully.)

(a) if the digits in x and y are the same everywhere beyond the n-th place (that is, $a_{n+1} = b_{n+1}$, $a_{n+2} = b_{n+2}$, $a_{n+3} = b_{n+3}, \ldots$) then

$$c_0.c_1c_2\ldots c_n = (a_0.a_1a_2\ldots a_n) - (b_0.b_1b_2\ldots b_n).$$

(b) Otherwise, let k be the smallest positive integer such that $a_{n+k} \neq b_{n+k}$. Then

$$c_0.c_1c_2\ldots c_n = (a_0.a_1a_2\ldots a_n\ldots a_{n+k}) - (b_0.b_1b_2\ldots b_n\ldots b_{n+k}), \text{ chopped off at the } n\text{-th decimal place.}$$

Now you have to convince yourself that each of the finite decimals $c_0.c_1c_2\ldots c_n$ agrees with the next one $c_0.c_1c_2\ldots c_nc_{n+1}$ up to the n-th place; and so all of the finite decimals can be merged together into a single infinite one.

For example, find the first four digits of .12344445 - .11144447. If you follow the recipe for $n = 3$, you'll see why you couldn't just round off to four places to find the first three decimal places of the difference—you have to go all the way out to the eighth decimal place (that is, take $k = 5$) in order to get the first three exactly right.

BOTTOM LINE. Now that you know how to subtract, you can check that, as stated above,

$$1.00000\cdots - 0.99999 = 0.00000\ldots.$$