MA 174 Final Exam May 2, 1988 Name:______________________________

(10 points) Find the directional derivative of the function $F(x, y, z) = e^x \sin(yz)$ at the point $(0, 1, \pi)$ in the direction of most rapid increase.

(15 points)

(a) Write an equation for the line passing through the points $(1, 2, 3)$ and $(-1, 1, 0)$.

(b) Write an equation for the plane containing the points $(1, -1, 2)$, $(2, 1, 1)$, and $(0, 1, 2)$.

(c) Find the angle between the planes given by the equations $2x - y = -2$ and $x - 2z = 1$.

(10 points) Let $\mathbf{r}(t) = (e^t \sin t, e^t \cos t)$ for $0 \le t \le 2$. What is the length of this curve?

(15 points) Consider the graph $y = \ln x$ for $x > 0$. Recall that the curvature $k(x)$ of the curve $y = f(x)$ is given by the formula

$$k(x) = \frac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}}.$$

(a) Show that $k(x) = x/(1 + x^2)^{3/2}$.

(b) Find the point on the curve for which the curvature is maximal. What is the curvature there?

(10 points) Find the tangent plane to the surface given by the formula

$$x^2y^2 + e^{yz} = 2$$

at the point $(1, 1, 0)$.

(15 points) Show that for any two vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\|\mathbf{a} + \mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2 = 4\,\mathbf{a} \cdot \mathbf{b}.$$

(Hint: $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$.)

(20 points) Find the critical points of the function $f(x, y) = (x^3 - 3x)(1 + y^2)$ and classify each critical point as a maximum, minimum, or saddle point.

(15 points) Let U be the sphere $x^2 + y^2 + z^2 \leq 1$, let S be the boundary surface of U, and let $\mathbf{n}$ be the outward normal to S. Let $\mathbf{v}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Evaluate $\iint_S \mathbf{v} \cdot \mathbf{n}\, d\sigma$.

(10 points) Let T be the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + 2y + 3z = 6$.

(a) Carefully sketch the region T below.

(b) Fill in the boxes; do not evaluate the integral.

$$\text{Volume of } T = \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \boxed{}\, dx\, dy\, dz.$$

(20 points) Find the centroid of the solid bounded above by the surface $z-(x^2+y^2) = 0$, below by the plane $z = 0$, and at the sides by the cylinder $x^2 + y^2 = 1$. Is the centroid inside the solid? Justify your answer. (Hint: Sketching the solid may help reduce the computation.)

(15 points) An object travels along the ellipse $x = a\cos t,\ y = b\sin t$ for $0 \le t \le 2\pi$ subject to a force $\mathbf{F}(x,y) = -\frac{1}{2}[y\,\mathbf{i} - x\,\mathbf{j}]$.

(a) Write the work done by the force as a line integral.

(b) Express the work done by the force in terms of the area of the ellipse.

(20 points) Calculate the line integral of the vector field

$$\mathbf{F}(x,y) = \frac{x}{(x^2+y^3)^{1/2}}\,\mathbf{i} + \frac{3y^2}{2(x^2+y^3)^{1/2}}\,\mathbf{j}$$

along the circular arc $x^2 + y^2 = 2$ from the point $(1,1)$ counter-clockwise to the point $(0,\sqrt{2})$.

(25 points) Let the surface S be the upper hemisphere given by the formula $x^2+y^2+z^2 = 4$ for $z \geq 0$. Let $\mathbf{n}$ be the upper unit normal of S, and let the curve C be the boundary of S. Let $\mathbf{v}(x,y,z) = -y\,\mathbf{i} + x\,\mathbf{j}$. Calculate

$$\iint_S \operatorname{curl} \mathbf{v} \cdot \mathbf{n} \, d\sigma$$

(a) as a surface integral.
(b) as a line integral using Stokes' theorem.