Values of L-series at negative integers

Recall $\zeta(1-2n) = -\dfrac{B_{2n}}{2n}$

where $\dfrac{t}{e^t-1} = \sum_{k=0}^{\infty} \dfrac{B_k t^k}{k!}$.

Thm Let $\varphi(s) = \sum \dfrac{a_n}{n^s}$ converge value of s, and let

$$f(t) = \sum a_n e^{-nt}$$

Suppose there is a sequence $b_{-1}, b_0, b_1, \dots$ s.t for all integers $N \geq 0$,

$$f(t) - \left(\frac{b_{-1}}{t} + b_0 + b_1 t + \dots + b_{n-1} t^n\right) = O(t^N)$$

(e.g., if f is meromorphic, with at most simple pole, at 0)

Then $\varphi(s)$ extends meromorphically to the whole plane, $\varphi(s) - \dfrac{b_{-1}}{s-1}$ is holomorphic, and

$$\varphi(-n) = (-1)^n n!\, b_n. \qquad n = 0, 1, 2, \dots$$

Example $a_n = 1 \ \forall n$ $\varphi(s) = \zeta(s)$

$$f(t) = \sum e^{-nt} = \frac{1}{e^t - 1} = \frac{1}{t} \sum_0^{\infty} \frac{B_k}{k!} t^k = \frac{1}{t} + \sum_0^{\infty} \frac{B_{n+1}}{(n+1)!} t^n$$

$$\Rightarrow \zeta(s) - \frac{1}{s-1} \text{ holomorphic in } \mathbb{C}$$

$$\zeta(-n) = (-1)^n n! \frac{B_{n+1}}{(n+1)!} = (-1)^n \frac{B_{n+1}}{n+1}$$

Proof Where $\varphi(s)$ converges absolutely, we have

$$\Gamma(s)\,\varphi(s) = \int_0^\infty f(t)\,t^{s-1}\,dt \qquad \text{(proved before)}$$

$$= I_1(s) + I_2(s)$$

$$I_1(s) = \int_0^1 f(t)\,t^{s-1}\,dt \qquad I_2(s) = \int_1^\infty f(t)\,t^{s-1}\,dt$$

Since $f(t) = O(e^{-t})$, $\therefore I_2(s)$ cvges for all s, absolutely and uniformly on compacta, hence is holomorphic.

Now
$$\int_0^1 \Big(\sum_{n<N} b_n t^n\Big) t^{s-1}\,dt = \sum_{n<N} b_n \frac{t^{n+s}}{n+s}\Big|_0^1$$

$$= \sum_{n<N} \frac{b_n}{n+s} \qquad |\sigma > 1|$$

$$\Rightarrow I_1(s) = \sum_{-1\le n<N} \frac{b_n}{n+s} + \int_0^1 \Big(f(t) - \sum_{n<N} b_n t^n\Big) t^{s-1}\,dt$$

integral converges for $\sigma > -N$, absolutely & uniformly on compacta, $\therefore$ holomorphic

So $\Gamma(s)\varphi(s) - \sum_{n<N} \frac{b_n}{n+s}$ extends holomorphically in $\sigma > -N$

Hence $\Gamma(s)\varphi(s)$ has meromorphic extension, with poles at most at $s = 1, 0, -1, -2, \dots$, with residue b_n at $s = -n$

Since $\frac{1}{\Gamma(s)}$ is holomorphic, with zero at $s = 0$, $s = -1, \dots$ and value 1 at $s = 1$

$\therefore \varphi(s)$ has meromorphic extension, pole at 1 with residue b_{-1}, ~~also~~ residue $\rho(n)$ of $\Gamma(s)$ at $s = -n$ $(n \geq 0)$

satisfies

$$\rho(n) = \lim_{s \to -n} (s+n)\Gamma(s) = \lim_{s \to -n} \frac{(s+n)\Gamma(s+1)}{s}$$

$$(\text{put } t = s+1) \qquad = \lim_{t \to 1-n} \frac{(t+n-1)\Gamma(t)}{t-1}$$

$$= \frac{-1}{n}\rho(n-1)$$

now $\rho(0) = \lim_{s \to 0} s\Gamma(s) = \lim_{s \to 0} \Gamma(s+1) = \Gamma(1) = 1$

$\Rightarrow f(n) = \frac{(-1)^n}{n!}$ So $\varphi(-n) = (-1)^n n!\, b_n$

Next, let $\chi \pmod N$ be a character, $a_n = \chi(n)$

$\varphi(s) = L(s, \chi)$.

$$f(t) = \sum_{n=1}^{\infty} \chi(n) e^{-nt}$$

$$= \sum_{m=1}^{N} \chi(m) \left(e^{-mt} + e^{-(m+N)t} + e^{-(m+2N)t} + \cdots\right)$$

$$= \sum_{m=1}^{N} \chi(m) \frac{e^{-mt}}{1-e^{-Nt}}$$

$$= \sum_{m=1}^{N} \chi(m) \sum_{k=0}^{\infty} (-m)^k \frac{t^k}{k!} \sum_{r=0}^{\infty} (-1)^r \frac{B_r}{r!} (Nt)^{r-1}$$

$$= \sum_{m=1}^{N} \chi(m) \sum_{k,r=0}^{\infty} (-1)^{k+r} \frac{m^k N^{r-1} B_r}{r!\,k!} t^{r+k-1}$$

$$= \sum b_n t^n$$

$$b_n = \sum_{m=1}^{N} \chi(m) \sum_{\substack{k,r \geq 0 \\ k+r = n+1}} (-1)^{k+r} \frac{B_r m^k N^{r-1}}{k!\,r!} \qquad n \geq -1$$

In particular, $b_{-1} = \frac{1}{N} \sum_{m=1}^{N} \chi(m) = 0$ if $\chi \neq \chi_0$

$\chi \neq \chi_0$ So $L(s, \chi)$ has holomorphic extension to $\mathbb{C}$,

$L(s, \chi_0)$ has meromorphic extension with pole at $s=1$ having residue $\varphi(N)/N$.