Formula for abscissa of convergence

Thm The abscissa of convergence of $\sum_{n=1}^{\infty} a_n n^{-s}$ is

$$\sigma_0 = \limsup \frac{\log |A'_N|}{\log N} = \inf\{\alpha \mid |A'_N| = O(N^\alpha)\} =: \gamma$$

where $A'_N = A(N)$ if $\sum a_n$ diverges $(\Rightarrow \sigma_0 \geq 0)$

and $A'_N = \sum a_n - A(N)$ if $\sum a_n$ converges $(\Rightarrow \sigma_0 \leq 0)$

and $|A'_N| = O(N^\alpha)$ means, $\exists\, C$ s.t. $|A'_N| \leq C(N^\alpha)$ for all N.

Remark

Proof ~~Note that $\gamma \geq 0$ unless $\sum a_n$ ... (since N^α ...)~~ If $\sum a_n$ cvges then $A'_N \to 0$ and so $\gamma \leq 0$. If $\sum a_n$ diverges then $\gamma \geq 0$. Let's justify the = sign in the Thm. If $\alpha < \gamma$, then, ~~for some ..., ... $< \gamma$, and so~~ $|A'_N|/N^\alpha$ is an unbounded sequence, $\Rightarrow |A'_N| > N^\alpha$ for infinitely many N

$\Rightarrow \dfrac{\log |A'_N|}{\log N} > \alpha$ for infinitely many N

$\Rightarrow \alpha \leq \limsup \dfrac{\log |A'_N|}{\log N}$. Hence $\gamma \leq \limsup \dfrac{\log |A'_N|}{\log N}$

Conversely, if $\alpha > \gamma$ then for some $\epsilon > 0$, $\alpha > \gamma + \epsilon$,

$\Rightarrow |A'_N| < CN^{\alpha - \epsilon} < N^\alpha$ for all $N >> 0$,

$\Rightarrow \dfrac{\log |A'_N|}{\log N} < \alpha$ for all $N >> 0$

$\Rightarrow \alpha \geq \limsup \dfrac{\log |A'_N|}{\log N}$ $\Rightarrow \gamma \geq \limsup \dfrac{\log |A'_N|}{\log N}$

Proof of formula for cvgce abscissa of $\sum_{n=1}^{\infty} a_n n^{-s}$.

Now suppose $\sigma < \gamma$, so that $|A'_N|/N^\sigma$ is unbounded.

I want to show that $\sigma \le \sigma_0$. S'pose not. Then $\sum a_n n^{-\sigma}$ converges ~~converges~~ and so $B(N) := \sum_1^N a_n n^{-\sigma}$ satisfies $|B(N)| < C$ for some constant C, and all N.

If $\sum a_n$ diverges, so that $\sigma > \sigma_0 \ge 0$ and $A'_N = A(N)$, then summation by parts gives

$$|A(N)| = \left|\sum_{n=1}^{N} a_n n^{-\sigma} n^{\sigma}\right| = \left|\sum_{n=1}^{N-1} B(n)(n^\sigma - (n+1)^\sigma) + B(N)N^\sigma\right|$$

$$\le \sum_{n=1}^{N-1} |B(n)|\left((n+1)^\sigma - n^\sigma\right) + |B(N)N^\sigma|$$

$$< C\sum_{n=1}^{N-1}\left((n+1)^\sigma - n^\sigma\right) + CN^\sigma \le 2CN^\sigma,$$ so that

$\gamma \le \sigma$, contradiction. Thus $\gamma \le \sigma_0$

If $\sum a_n$ converges, say to S, then $\gamma \le 0$, and $\sigma < 0$.

So, now get (as above) $|A'_{n+1}| \le C\left[n^\sigma + n^\sigma - (n+1)^\sigma + (n+1)^\sigma - (n+2)^\sigma \ldots\right.$

$$= 2Cn^\sigma \quad (\text{since } \sigma < 0 \Rightarrow n^\sigma \to 0)$$

and, again, $\gamma \le \sigma$, contradiction.

Conclude $\gamma \le \sigma_0$

Next suppose $\sigma > \gamma$. If $\sum a_n$ diverges, then $\gamma \ge 0$ and $\sigma > 0$. Then summation by parts gives

$$B(N) = \sum_{n=1}^{N-1} A(n)(n^{-\sigma} - (n+1)^{-\sigma}) + A(N)N^{-\sigma}$$

Similar argument when $\sum a_n$ converges, using $a_n = A'_n - A'_{n+1}$ and $|n^{-\sigma} - (n+1)^{-\sigma}| = \sigma(n+\varepsilon)^{-\sigma-1}$ $0 < \varepsilon < 1$.

Choose α with $\gamma < \alpha < \sigma$, $|A(N)| < CN^\alpha$ $\forall N$. So

$$|A(n)(n^{-\sigma} - (n+1)^{-\sigma})| \le Cn^\alpha(n^{-\sigma} - (n+1)^{-\sigma}) < C\sigma n^{\alpha-\sigma-1} \quad (\text{MVT})$$

and $|A(N)N^{-\sigma}| \le CN^{\alpha-\sigma} \to 0$. Also $\sum_{n=1}^{\infty} n^{\alpha-\sigma-1} < \infty \Rightarrow$

$\lim_{n\to\infty} |B(M) - B(N)| \to 0$ for large M, N $\Rightarrow \lim_{N\to\infty} B(N) = \sum a_n n^{-\sigma}$ exists

$\therefore \sigma \ge \sigma_0$; and thus $\gamma \ge \sigma_0$.