Compact operators

Definition. A bounded linear operator A on a Hilbert space $\mathcal{H}$ is *compact* if it has any of the following properties:

(1) There exists a sequence $F_1, F_2, \dots$ of finite rank operators such that $\|F_n - A\|_{\mathcal{H}\to\mathcal{H}} \to 0$ as $n \to \infty$.
(2) For any bounded sequence $w_1, w_2, \dots$ in $\mathcal{H}$, the sequence $Aw_1, Aw_2, \dots$ has a convergent subsequence.
(3) For any sequence $w_1, w_2, \dots$ converging weakly to any w, we have $\|A(w_n - w)\|_{\mathcal{H}} \to 0$.

Theorem. *The properties (1), (2), and (3) are all equivalent.*

Proof. (1) $\Rightarrow$ (2): Use a diagonal argument to construct a subsequence $w_{11}, w_{22}, \dots$ such that $F_n w_{11}$, $F_n w_{22}, \dots$ converges for each n. To prove that $Aw_{11}, Aw_{22}, \dots$ is Cauchy, write

$$\|Aw_{kk} - Aw_{\ell\ell}\| \le \|(A - F_n)w_{kk}\| + \|F_n(w_{kk} - w_{\ell\ell})\| + \|(A - F_n)w_{\ell\ell}\|.$$

Given $\varepsilon > 0$, first choose n large enough that the first and last terms on the right are less than $\varepsilon/3$, and then choose M large enough that the middle term is less than $\varepsilon/3$ when $k, \ell \ge M$.

(2) $\Rightarrow$ (3): A weakly convergent sequence is bounded by the Uniform Boundedness Principle, so $Aw_1, Aw_2, \dots$ has a convergent subsequence. But any convergent subsequence of $Aw_1, Aw_2, \dots$ must converge to Aw, since it converges weakly to Aw. Hence $Aw_1, Aw_2, \dots$ converges to Aw.

(3) $\Rightarrow$ (1): Define an orthonormal sequence $e_1, e_2, \dots$ in $\mathcal{H}$ as follows. First, take e_1 such that

$$\|Ae_1\|_{\mathcal{H}} \ge \tfrac{1}{2}\|A\|_{\mathcal{H}\to\mathcal{H}}.$$

Then proceed inductively: having defined $e_1, \dots, e_n$, take e_{n+1} such that

$$\|Ae_{n+1}\|_{\mathcal{H}} \ge \tfrac{1}{2}\|A(I - P_n)\|_{\mathcal{H}\to\mathcal{H}},$$

where P_n denotes orthogonal projection onto the span of $\{e_1, \dots, e_n\}$. Since $e_1, e_2, \dots$ converges weakly to 0, it follows that $\|Ae_n\|_{\mathcal{H}} \to 0$, and hence $\|A(I - P_n)\|_{\mathcal{H}\to\mathcal{H}} \to 0$ and we may take $F_n = AP_n$. □

The above proof follows Section 3.1 of [Sim]. That reference also covers compact operators on a Banach space, as does [Con]. For a general Banach space one has only (1) $\Rightarrow$ (2) $\Rightarrow$ (3). If the space is reflexive, then (3) $\Rightarrow$ (2). If it has a Schauder basis, then (2) $\Rightarrow$ (1).

References

[Con] John B. Conway, *A Course in Functional Analysis*, Second Edition, 1990.
[Sim] Barry Simon, *Operator Theory: A Comprehensive Course in Analysis, Part 4*, 2015.

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