Fréchet spaces

The concept of Hilbert space generalizes and systematizes the geometric structure of Euclidean spaces and L^2 spaces. The concept of Banach space takes the generalization a step further, including such spaces as continuous and differentiable functions on a compact set and L^p spaces.

The concept of Fréchet space goes yet further, allowing one to study functions on open sets; this is important because many properties of functions are better on open sets, for instance complex differentiable functions are analytic. The price to pay is that at each step of generalization we encompass nastier spaces, and the abstract results are correspondingly more complicated and weaker.

These notes develop the properties of Fréchet space just far enough to be able to understand Example 2.4.3 of [Hör]. That example gives a necessary condition for a constant-coefficient differential operator to improve regularity. The condition is also sufficient but a lot more is involved in the proof of sufficiency: see Theorem 6.36 of [Fol] or Theorem III.2.1 of [Tay].

Below, we start with the basic definitions, followed by examples and exercises with hints. The basic definition is that a Fréchet space is a complete metric space with the metric defined by a family of seminorms, a seminorm being a slightly weaker version of a norm. An important feature is that continuous functions are characterized by a boundedness condition. We conclude with the Open Mapping Theorem, which is the big machine used in Example 2.4.3 of [Hör].

See Section V.2 of [ReSi] for more on Fréchet spaces, or Sections 2.2 and 2.4 of [Hör] for even more.

Definitions. Let X be a vector space over $\mathbb{F}$. A function $\|\cdot\|\colon X \to [0,\infty)$ is called a *seminorm* if it obeys

$$\|x+y\| \le \|x\| + \|y\|, \qquad \|cx\| = |c|\|x\|,$$

for all $x, y \in X$, $c \in \mathbb{F}$; i.e. the same properties as a norm except we allow $\|x\| = 0$ for $x \neq 0$.

Let $(\|\cdot\|_k)_{k=1}^\infty$ be a sequence of seminorms on X which separates points, in the sense that

$$\|x\|_k = 0 \text{ for all } k \Longrightarrow x = 0.$$

It is convenient to suppose further that for all k and x we have

$$\|x\|_k \le \|x\|_{k+1}.$$

(The general case can be reduced to this one by defining $\|x\|_k' = \max_{1\le j\le k} \|x\|_j$.) Define a metric on X by putting

$$d(x,y) = d(x-y), \qquad d(x) = \sum_{k=1}^\infty 2^{-k} \frac{\|x\|_k}{1+\|x\|_k}; \tag{1}$$

to prove the triangle inequality, prove

$$f(t) = \frac{t}{1+t} \Longrightarrow f(t) \le f(t+s) \le f(t) + f(s) \text{ for all } t, s \ge 0,$$

by clearing denominators, and deduce $d(x+y) \le d(x) + d(y)$.

If (X,d) is complete, then we say that (X,d) is a *Fréchet space*.

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Examples. Let $\Omega \subset \mathbb{R}^d$ be open, and let $K_1 \subset K_2 \subset \dots$ be compact subset of Ω such that $\Omega = \bigcup_{k=1}^\infty K_k$. For example, we may let

$$K_k = \{x \in \Omega \colon |x| \le k \text{ and } \operatorname{dist}(x, \partial\Omega) \ge 1/k\}.$$

The following are all Fréchet spaces.

1. Let $X = C(\Omega)$, the set of continuous functions on Ω, and let
$$\|u\|_k = \max_{K_k} |u|.$$
2. Let $X = C^m(\Omega)$, the set of m times continuously differentiable functions on Ω, and let
$$\|u\|_k = \max_{|\alpha| \le m} \max_{x \in K_k} |\partial^\alpha u(x)|.$$
3. Let $X = C^\infty(\Omega)$, the set of infinitely differentiable functions on Ω, and let
$$\|u\|_k = \max_{|\alpha| \le k} \max_{x \in K_k} |\partial^\alpha u(x)|.$$

Exercises.

1) Let (X, d) be a metric vector space as above, and let $x_1, x_2, \dots$ be a sequence in X. Prove that $d(x_j) \to 0$ if and only if $\|x_j\|_k \to 0$ for all k.
2) Let X and Y be metric vector spaces as above, and let $T \colon X \to Y$ be linear. Prove that T is continuous if and only if for any k there are k' and C such that
$$\|Tx\|_k \le C\|x\|_{k'}.$$
3) Let $(X, \|\cdot\|)$ be a Banach space, and define d by (1) with $\|\cdot\|_k = \|\cdot\|$ for all k, i.e. all the seminorms are just the Banach space norm over and over. Prove that the resulting (X, d) is a Fréchet space with the same convergent sequences as $(X, \|\cdot\|)$.

Hints: Prove that $d(x) \le \|x\|_k + 2^{-k}$ for any k and that $d(x) \le 2^{-k-\ell} \implies \|x\|_k \le 2^{-\ell}/(1 - 2^{-\ell})$ for any k and ℓ. Deduce 1), and then use 1) to prove 2) and 3); proving 2) is similar to proving that a linear function between normed vector spaces is continuous if and only if it is bounded.

Theorem 1 (Baire Category Theorem)**.** *If X is a complete metric space and $\bigcup_{n=1}^\infty A_n = X$, then there is n such that $\overline{A_n}$ has nonempty interior.*

This is proved in many places; an elegant version is Theorem 9.1 of [Oxt]. We use it to prove:

Theorem 2 (Open Mapping Theorem)**.** *If X and Y are Fréchet spaces, and if $T \colon X \to Y$ is a surjective continuous linear function, then $T(U) \subset Y$ is open for every open $U \subset X$.*

This is also proved in many places. The proof below is similar to that of Theorem III.10 of [ReSi] and Theorem II.5 of [Yos]. See also Section 2.4 of [Hör].

Proof. We use the notation $X_r(x) = \{x' \in X \colon d_X(x' - x) < r\}$, $X_r = X_r(0)$, and similarly for Y.

(1) Since open sets are made up of open balls, it is enough to show that for any $r > 0$ and $x \in X$, there is $r' > 0$ such that
$$Y_{r'}(y) \subset T(X_r(x)).$$

(2) Since T is linear, it is enough to show that for any $r > 0$, there is $r' > 0$ such that
$$Y_{r'} \subset T(X_r).$$

(3) We claim that it is enough to show that $T(X_r)$ has nonempty interior for any $r > 0$.

Exercise. Let X and Y be Fréchet spaces, let $T \in \mathcal{L}(X,Y)$, let $y \in Y$, and let $r, r' > 0$. Show that if $Y_{r'}(y) \subset T(X_r)$, then $Y_{r'} \subset T(X_{2r})$.

(4) We claim that $\overline{T(X_r)}$ has nonempty interior for any $r > 0$. Indeed, given $r > 0$, $X = \bigcup_{n=1}^{\infty} nX_r$, because, for any $x \in X$, $d_X(x/n) \leq 1/n \to 0$ as $n \to \infty$. Since T is surjective, $Y = \bigcup_{n=1}^{\infty} T(nX_r)$. By the Baire category theorem, $\overline{T(nX_r)}$ has nonempty interior for some n. But
$$\overline{T(nX_r)} = \overline{nT(X_r)} = n\overline{T(X_r)},$$
where the first equals follows from linearity of T, and the second from the fact that $y_j \to y$ if and only if $ny_j \to ny$, which in turn follows from $d(y_j) \to 0$ if and only if $\|y_j\|_K \to 0$ for all K. Hence $\overline{T(X_r)}$ has nonempty interior.

(5) The final step is showing that $\overline{T(X_r)} \subset T(X_{2r})$, and hence that $T(X_{2r})$ has nonempty interior. Take $y \in \overline{T(X_r)}$. We will find $x_1 \in X_r$, $x_2 \in X_{r/2}$, ..., $x_j \in X_{r/2^j}$, ..., such that
$$\lim_{N\to\infty} d_Y(y - s_N) = 0,$$
where $s_N = x_1 + \cdots + x_N$. Since X is complete, and $d(s_N - s_M) < 2^{-N}r$ when $M \geq N$, it follows that there is x such that $d_X(x - s_N) \to 0$. Since T is continuous, $Tx = y$.

It remains to construct the x_j. For this, take a sequence $r'_j \to 0$ such that
$$Y_{r'_j} \subset \overline{T(X_{r/2^j})};$$
such an r'_j exists for each j because each $\overline{T(X_{r/2^j})}$ has nonempty interior and we can proceed as in the Exercise from Step (3). Now:

(i) Take $x_1 \in X_r$ such that $y - Tx_1 \in Y_{r'_1} \subset \overline{T(X_{r/2})}$.

(ii) Take $x_2 \in X_{r/2}$ such that $y - Ts_2 \in Y_{r'_2} \subset \overline{T(X_{r/4})}$.

(iii) Continue in this manner, taking $x_k \in X_{r/2^k}$ such that $y - Ts_k \in Y_{r'_k} \subset \overline{T(X_{r/2^k})}$.

We have arranged $d_Y(y - s_N) < r'_N$, which tends to 0 as $N \to \infty$, as desired.

□

References

[Fol] Gerald B. Folland, *Introduction to Partial Differential Equations*, 1995.
[Hör] Lars Hörmander, *Linear Functional Analysis*, 1988.
[Oxt] John C. Oxtoby, *Measure and Category*, 1980.
[ReSi] Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, 1980.
[Tay] Michael E. Taylor, *Pseudodifferential Operators*, 1981.
[Yos] Kôsaku Yosida, *Functional Analysis*, 1970.