Kiril Datchev
MA 546
Spring 2025

Homework 9

Due March 26th on paper at the beginning of class. Please let me know if you have a question or find a mistake.

1. Let K be a compact metric space, and T a continuous map $K \to K$. Prove that there exists a T-invariant Borel probability measure $d\mu$ on K by following these steps:
   (a) Let $d\mu_1$, $d\mu_2$, ... be a sequence of Borel probability measures on K. Prove that there exist a subsequence $d\mu_{n_k}$ and a Borel measure $d\mu$ such that $\int f\, d\mu_{n_k} \to \int f\, d\mu$ for all $f \in C(K)$.
   (b) Specialize part (a) to the case $d\mu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{x_j}$, where each x_j is a point in K. Prove that then $d\mu$ is a probability measure.
   (c) Specialize part (b) to the case where $x_{n+1} = Tx_n$ for $n \geq 1$. (The initial point x_1 can be any point in K.) Prove that then $\int f\, d\mu = \int (f \circ T)\, d\mu$ for all $f \in C(K)$.
2. Consider $Tu(x) = \int_0^x u$ on $L^2(0,1)$. For $\lambda \neq 0$, give an explicit formula for $(T - \lambda)^{-1}$. Use this to find $\rho(T)$ and $\sigma(T)$.
3. Exercise 4.10 from *Borthwick*.

See the back page for some hints.

Hints:

1. (a) Use the Riesz representation theorem, in the form which says that we can identify Borel measures on K with linear functionals on $C(K)$ which are nonnegative on nonnegative functions. Let $f_0, f_1, \ldots$ be a dense sequence in $C(K)$, and use a diagonal argument to get the desired convergent subsequence.
   (b) Evaluate μ on constant functions.
   (c) Look at the differences $\int f \, d\mu_n - \int (f \circ T) \, d\mu_n$.
2. Solve $Tu - \lambda u = f$ by substituting $Tu = v$.