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Arbeitsgemeinschaft: Geometric Representation Theory

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Abstract. Our understanding of algebraic representations of reductive algebraic groups in positive characteristic has seen big advances in the last years and has been largely transformed into the geometric theory of studying parity sheaves on affine Grassmannians and affine flag varieties or, equivalently and more combinatorially, the diagrammatic Hecke category. This has led, among other things, to a geometric proof of the linkage principle and a greatly simplified proof of Lusztig's character formula for large characteristics.

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Introduction by the Organizers

The Arbeitsgemeinschaft *Geometric Representation Theory*, organised by Daniel Juteau, Simon Riche, Wolfgang Soergel and Geordie Williamson, attracted excellent researchers of various backgrounds from all over the world, including many graduate students and postdocs. Due to the Corona virus restrictions it was organised as a hybrid event with 33 real and 23 virtual participants. Also one of the organisers was present only virtually due to travel restrictions and another could not come. As usual for an Arbeitsgemeinschaft, the organisers provided a detailed program and distributed the talks to the participants. We had a total of 18 talks of one hour each with ample time for discussion and additional discussion sessions from eight to ten in the evenings. On Wednesday afternoon, we made an excursion to St. Roman to get some Schwarzwälder Kirschtorte and on Thursday evening, after the discussion and decision on the program of the Arbeitsgemeinschaft in a year to come moderated by Peter Scholze, we organised a musical event and get-together.

The program started out with a discussion of the general framework of highest weight categories and Kazhdan–Lusztig polynomials, and explained the significance of the model case of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$, as studied in particular using Beĭlinson–Bernstein localization theory. This was followed by a discussion of algebraic representations of reductive algebraic groups and its main features, in particular the linkage principle and translation functors. In a second part we learned about categorifications of the Hecke algebra via categories of constructible sheaves on (affine) flag varieties, culminating with the introduction of the diagrammatic Hecke category. After that, in a third part, Lusztig's conjectural character formula for simple representations in the principal block was discussed, in particular from the point of view of the (later) Finkelberg–Mirković conjecture proposing a geometric incarnation of the principal block. This part finished with a presentation of the counterexamples to Lusztig's character formula found a few years ago by Williamson. In a fourth part, we presented the relation between the study of characters of simple and indecomposable tilting representations, and an approach to the latter question via an action of the Hecke category proposed by Riche–Williamson. Finally, the last part was devoted to the new geometric understanding of algebraic representations of reductive algebraic groups in positive characteristic offered by Smith–Treumann theory, applied in the context of the geometric Satake equivalence.

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Arbeitsgemeinschaft: Geometric Representation Theory

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Abstracts

Category $\mathcal{O}$ and Highest Weight Categories

Jeanine Van Order

This talk gave a rapid overview of the Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$, leading to the more general relevant notion of a highest weight category.

To fix ideas, let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field. The category Mod $\mathcal{U}(\mathfrak{g})$ of all $\mathcal{U}(\mathfrak{g})$-modules is too large to study algebraically. Bernstein-Gelfand-Gelfand [1], in their 1976 paper *On a category of* $\mathfrak{g}$*-modules*, introduced a convenient subcategory they called $\mathcal{O}$ – taken from the Russian word основной for "basic" – as a place to study Jordan-Hölder decompositions of Verma modules, and to establish an infinite-dimensional analogue of the Brauer-Nesbitt reciprocity theorem (BGG reciprocity). To be more precise, let us fix a Cartan subalgebra $\mathfrak{h} \subset \mathrm{g}$, together with system of positive roots $\Phi^+ \subset \Phi \subset \mathfrak{h}^*$, so that we have the corresponding Cartan decomposition

$$\mathfrak{g} \cong \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}.$$

Here,

$$\mathfrak{n} := \bigoplus_{\alpha>0} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}^- := \bigoplus_{\alpha<0} \mathfrak{g}_\alpha$$

for $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : [h,x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$, so that $\mathfrak{b} \cong \mathfrak{h} \oplus \mathfrak{n}$ describes the standard Borel subalgebra, and $\mathfrak{b}^- \cong \mathfrak{h} \oplus \mathfrak{n}^-$ its opposite. We then define $\mathcal{O} = \mathcal{O}(\mathfrak{g}, \mathfrak{h}, \Phi^+)$ to be the subcategory of all modules $M \in$ Mod $\mathcal{U}(\mathfrak{g})$ for which

- ($\mathcal{O}_1$) M is a finitely generated $\mathcal{U}(\mathfrak{g})$-module,
- ($\mathcal{O}_2$) M is a semisimple $\mathfrak{h}$-module, and hence a weight module $M \cong \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ (where each $M_\lambda := \{x \in M : \ hx = \lambda(h)x \ \forall h \in \mathfrak{h}\}$),
- ($\mathcal{O}_3$) M is locally $\mathfrak{n}$-finite, i.e. for all $x \in M$, the span $\mathcal{U}(\mathfrak{n})x$ is finite dimensional,

from which one can deduce via the Poincaré-Birkhoff-Witt (PBW) decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-)\mathcal{U}(\mathfrak{h})\mathcal{U}(\mathfrak{n})$ that

- ($\mathcal{O}_4$) Each weight space M_λ is finite dimensional,
- ($\mathcal{O}_5$) The set $\Pi(M) := \{\lambda \in \mathfrak{h}^* : M_\lambda \neq 0\}$ is contained in a finite union of sets of the form $\lambda - \Gamma$ for some $\lambda \in \mathfrak{h}^*$, where $\Gamma = \langle \Phi^+ \rangle$ denotes the span of the positive roots.

It is not hard to show the following basic properties (see [3, (1.1)] or [5, §3]):

- (a) $\mathcal{O}$ is noetherian, i.e. each $M \in \mathcal{O}$ is a noetherian $\mathcal{U}(\mathfrak{g})$-module.
- (b) $\mathcal{O}$ is closed under taking submodules, quotients, and finite direct sums.
- (c) $\mathcal{O}$ is an abelian category.
- (d) Given $M \in \mathcal{O}$ and L a finite dimensional $\mathcal{U}(\mathfrak{g})$-module, $L \otimes M \in \mathcal{O}$ (and hence $M \mapsto L \otimes M$ defines an exact functor $\mathcal{O} \to \mathcal{O}$).
- (e) Each $M \in \mathcal{O}$ is $Z(\mathfrak{g})$-finite, i.e. for any $x \in M$, the span $Z(\mathfrak{g})x$ is finite dimensional.
- (f) Each $M \in \mathcal{O}$ is a finitely generated $\mathcal{U}(\mathfrak{n}^-)$-module.

The prototype example of a module in $\mathcal{O}$ is that of a highest weight module. Given $M \in \operatorname{Mod} \mathcal{U}(\mathfrak{g})$ and $\lambda \in \mathfrak{h}^*$, a nonzero vector $v^+ \in M$ is said to be a *maximal (or primitive) vector of weight* λ if $v^+ \in M_\lambda$ and $\mathfrak{n}v^+ = 0$. A module $M \in \mathcal{O}$ is then said to be a *highest weight module of weight* λ if there exists a maximal vector $v^+ \in M$ of weight λ for which $M = \mathcal{U}(\mathfrak{g})v^+$. We recalled some of the nice properties satisfied by such modules ([3, (1.2)]), and in particular the fact that each nonzero $M \in \mathcal{O}$ admits a finite filtration $0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ whose successive quotients M_i/M_{i-1} are highest weight modules. The prototype example of a highest weight module in this sense is that of the Verma module $M(\lambda) = \Delta(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbf{C}_\lambda$, where $\mathbf{C}_\lambda$ denotes the one-dimensional $\mathfrak{b}$-module with trivial $\mathfrak{n}$-action determined by the weight $\lambda \in \mathfrak{h}^*$. Writing $L(\lambda)$ to denote the unique simple submodule of the Verma module $M(\lambda)$, we also saw the important theorem that each simple module $M \in \mathcal{O}$ is isomorphic to some such $L(\lambda)$, and moreover that $\dim \operatorname{Hom}_{\mathcal{O}}(L(\lambda), L(\mu)) = \delta_{\lambda\mu}$. That is, the modules $L(\lambda)$ with $\lambda \in \mathfrak{h}^*$ varying describe the simple modules in $\mathcal{O}$. Moreover, a given $L(\lambda)$ is finite dimensional if and only if the weight λ is dominant.

Although constrained by time, we also described the character and block decompositions of $\mathcal{O}$. Here, we first mentioned the theorem of Harish-Chandra giving the identification $Z(\mathfrak{g}) \cong S(\mathfrak{h})^W$, and parametrizing all characters $\chi : Z(\mathfrak{g}) \longrightarrow \mathbf{C}$ in terms of the weights: $\chi = \chi_\lambda$ for some (uniquely-determined) $\lambda \in \mathfrak{h}^*$. While the action of the centre $Z(\mathfrak{g})$ on a given $M \in \mathcal{O}$ is complicated to describe in general, we can consider for a given central character $\chi = \chi_\lambda$ the submodule

$$M^\chi := \{v \in M : (z - \chi(z))^n \cdot v = 0 \text{ for some } n = n(z) \in \mathbf{Z}_{>0}\}.$$

It is not hard to deduce that M admits a decomposition into such submodules

$$M = \bigoplus_{\substack{\chi = \chi_\lambda \\ \lambda \in \mathfrak{h}^*}} M^\chi.$$

Writing $\mathcal{O}_\chi$ for each central character $\chi = \chi_\lambda$ (determined by some $\lambda \in \mathfrak{h}^*$) to denote the subcategory of $\mathcal{O}$ with objects M^χ (with $M \in \mathcal{O}$ varying), we then have the corresponding decomposition

$$\mathcal{O} \cong \bigoplus_{\substack{\chi = \chi_\lambda \\ \lambda \in \mathfrak{h}^*}} \mathcal{O}_\chi = \bigoplus_{\lambda \in \mathfrak{h}*} \mathcal{O}_{\chi_\lambda}.$$

Moreover, each highest weight module of weight λ is contained in the subcategory $\mathcal{O}_{\chi_\lambda}$, and each $\mathcal{O}_\chi$ contains only finitely many simples/Vermas. On the other hand, $\mathcal{O}$ is also artinian, and hence each $M \in \mathcal{O}$ admits a composition series with simple quotients isomorphic to the various $L(\lambda)$, and with the multiplicity $[M : L(\lambda)]$ in each series independent of the choice of series. We mentioned the Grothendieck group $K(\mathcal{O})$ in this connection too, together with its relation to the formal character $\operatorname{ch}(M)$ of $M \in \mathcal{O}$. *Block decompositions* appear naturally at this point. Here, the underlying idea is find some better organization to study modules which fail to be semisimple via Ext functors. To describe this more precisely, if two nontrivial simple modules $M_1, M_2 \in \mathcal{O}$ can be extended nontrivially in the sense that there exists a nonsplit short exact sequence $0 \to M_i \to M \to M_j \to 0$

for $\{i, j\} = \{1, 2\}$, we put M_1 and M_2 in the same block. More generally, given simples $M, N \in \mathcal{O}$ contained in a finite sequence $M = M_1, M_2, \ldots, M_n = N$ with adjacent pairs contained in the same block, we put M and N in the same block. In connection with the central character decomposition described above, a standard proposition here shows that each subcategory $\mathcal{O}_\chi = \mathcal{O}_{\chi_\lambda}$ forms a block in this sense if the weight λ is integral. Some other standard facts were mentioned without proof, including the fact that $\mathcal{O}$ has enough projectives, and implicitly that each projective admits a standard filtration (leading to the statement of BGG reciprocity; see [3, (3.10), (3.11)]). Additionally, for the instructive special case of $\mathfrak{g} = \mathfrak{sl}_2$, we noted that the the principal class $\mathcal{O}_0 = \mathcal{O}_{\chi_0}$ has precisely five indecomposables: (i) the simple module $L(0)$ of weight 0, (ii) the simple Verma module $L(-2) = M(-2)$ of weight -2, (iii) the projective Verma module $M(0) = P(0)$ of weight 0, (iv) the twisted dual $M(0)^\vee = Q(0)$, and (v) the projective cover $P(-2) = Q(-2)$ (of the injective envelope $Q(-2)$ of weight -2). More generally, each subcategory $\mathcal{O}_{\chi_\lambda}$ with $\lambda \geq 0$ integral admits only these five indecomposables: (i) the simple $L(\lambda)$ of weight λ, (ii) the simple Verma module $L(-\lambda - 2) = M(-\lambda - 2)$ of weight $-\lambda - 2$, (iii) the projective Verma module $M(\lambda) = P(\lambda)$ of weight λ, (iv) the twisted dual $M(\lambda)^\vee = Q(\lambda)$, and (v) the projective cover $P(-\lambda - \mu) = Q(-\lambda - \mu)$ (of the injective envelope $Q(-\lambda - \mu)$ of weight $-\lambda - \mu$) (see [3, Proposition 3.12]).

At last, we gave a motivated introduction of the more general notion of a highest weight category, following [4, Appendix A]. Briefly, let k be any field, and $\mathcal{A}$ any finite-length, k-linear abelian category such that $\dim \operatorname{Hom}_{\mathcal{A}}(M, N) < \infty$ for any pair $M, N \in \mathcal{A}$. Write $\mathcal{S}$ to denote the set of isomorphism classes of irreducible objects in $\mathcal{A}$. We assume that this set $\mathcal{S}$ comes equipped with a partial ordering $\leq$. We also assume that for each element $s \in \mathcal{S}$, we have a simple object representative $L_s \in s$, together with objects $\Delta_s, \nabla_s \in \mathcal{S}$ and morphisms

$$\Delta_s \longrightarrow L_s, \quad L_s \longrightarrow \nabla_s.$$

Given a subset $\mathcal{T} \subset \mathcal{S}$, let $\mathcal{A}_\mathcal{T} \subset \mathcal{A}$ denote the Serre subcategory generated by objects L_t with $t \in \mathcal{T}$. We then put

$$\mathcal{A}_{\leq s} := \mathcal{A}_{\{t \in \mathcal{S} : t \leq s\}}, \quad \mathcal{A}_{< s} := \mathcal{A}_{\{t \in \mathcal{S} : t < s\}}.$$

Recall that a subset $\mathcal{T} \subset \mathcal{S}$ is an *ordered ideal* if $t \in \mathcal{T}$ and $s \in \mathcal{S}$ with $s \leq t$, then $s \in \mathcal{T}$. Equipped with this data, we say that $\mathcal{A}$ is a *highest weight category* if

- ($\mathcal{A}_1$) For all $s \in \mathcal{S}$, the set $\{t \in \mathcal{S} : t \leq s\}$ is finite.
- ($\mathcal{A}_2$) For all $s \in \mathcal{S}$, $\operatorname{Hom}_{\mathcal{A}}(L_s, L_s) = k$.
- ($\mathcal{A}_3$) For all $s \in \mathcal{S}$ and ordered ideals $\mathcal{T} \subset \mathcal{S}$ where $s \in \mathcal{T}$ is maximal, $\Delta_s \longrightarrow L_s$ is a projective cover in $\mathcal{A}_\mathcal{T}$, and $L_s \longrightarrow \nabla$ is an injective envelope in $\mathcal{A}_\mathcal{T}$.
- ($\mathcal{A}_4$) $\ker(\Delta_s \longrightarrow L_s), \operatorname{coker}(L_s \longrightarrow \nabla_S) \in \mathcal{A}_{< s}$.
- ($\mathcal{A}_5$) $\operatorname{Ext}^2(\Delta_s, \nabla_s) = 0$ for all $s, t \in \mathcal{S}$.

We then call $(\mathcal{S}, \leq)$ the *weight poset* of $\mathcal{A}$, the objects Δ_s *standard objects*, and the ∇_s *costandard objects*. We say that an object $M \in \mathcal{A}$ *admits a Δ-filtration* if there exists a finite filtration of M whose subquotients are standard. Similarly, we say that M *admits a ∇-filtration* if there exists a finite filtration of M whose subquotients are costandard. An object $M \in \mathcal{A}$ is said to be *tilting* if it admits both Δ- and ∇-filtrations. It can be shown that for all $s, t \in \mathcal{S}$,

$$\mathrm{Hom}_{\mathcal{A}}(\Delta_s, \nabla_t) = \begin{cases} k & \text{if } s = t \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathrm{Ext}^1(\Delta_s, \nabla_s) = \{0\}.$$

Although not immediate, it can also be shown that the (integral) Berstein-Gelfand-Gelfand category $\mathcal{O}$ is an example of a highest weight category, with weight poset corresponding to the set of integral weights (see [2, Example 3.3 (c)]). In brief, this comes down to verifying that $\mathrm{Ext}^1_{\mathcal{O}}(M(\lambda), M(\mu)) \neq 0$ implies $\mu < \lambda$. (Note that $\mathrm{Ext}^1_{\mathcal{O}}(M(\lambda), M(\mu)) \neq 0$ implies $\mathrm{Ext}^1_{\mathcal{U}(\mathfrak{g})}(M(\lambda), M(\mu)) \cong \mathrm{Ext}^1_{\mathcal{U}(\mathfrak{b})}(\lambda, M(\mu)) \neq 0$. If $v \notin M(\mu)$ is a maximal vector of weight λ in a nonsplit extension of $M(\lambda)$ by $M(\mu)$, it follows that $e_\alpha v = 0$ for some positive root vector e_α, from which it follows that $\lambda + \alpha$ is a weight in $L(\mu)$, and hence that $\mu < \lambda$, as desired).

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Coxeter Groups and Kazhdan–Lusztig Polynomials

Alessio Cipriani

A Coxeter system is a pair (W, S) where W is a group that admits a presentation

$$W = \langle s \in S \mid (s_1 s_2)^{m(s_1, s_2)} = e \quad \forall s_1, s_2 \in S \rangle,$$

with

$$\begin{cases} m(s, s) = 1 & \forall s \in S \\ 2 \leq m(s_1, s_2) \leq \infty & \forall s_1 \neq s_2 \end{cases}.$$

The group W is called Coxeter group and elements $s \in S$ are called simple reflections. Any element $w \in W$ can be written as an expression $w = s_1 \dots s_k$ with $s_i \in S$ and an expression is said to be reduced if k is minimal. This allows one to define a length function $\ell : W \to \mathbb{N}$ which associates to each element $w \in W$ the length of a reduced expression and to introduce the Bruhat order $\leq$ on W.

Let $\mathcal{L} = \mathbb{Z}[v^{\pm 1}]$ be the ring of Laurent polynomials with integer coefficients in the variable v and (W, S) a Coxeter system. On the free $\mathcal{L}$-module with basis indexed by elements in W

$$\mathcal{H} = \mathcal{H}(W, S) = \bigoplus_{x \in W} \mathcal{L} T_x$$

there is a unique associative algebra structure given by the relations

$$\begin{cases} T_x T_y = T_{xy} & \text{if} \quad \ell(x) + \ell(y) = \ell(xy) \\ T_s^2 = v^{-2} T_e + (v^{-2} - 1) T_s & \forall s \in S \end{cases},$$

called the Hecke algebra corresponding to (W, S).

For any $w \in W$ we set $H_w = v^{\ell(w)} T_w$, so that $H_e = 1$ and $H_s = vT_s$ for any $s \in S$. With this substitution, the relations imposed on $\mathcal{H}$ become the quadratic relations $H_s^2 = 1 + (v^{-1} - v) H_s$ if $s \in S$ as well as braid relations. Moreover, there is exactly one ring homomorphism, called bar involution,

$$\begin{aligned} \overline{} \colon \mathcal{H} &\to \mathcal{H} \\ H &\mapsto \overline{H} \end{aligned}$$

which sends $v \mapsto \overline{v} = v^{-1}$ and $H_x \mapsto \overline{H}_x = H_{x^{-1}}^{-1}$. One says that an element $H \in \mathcal{H}$ is self-dual if it is fixed by the above involution, that is if $H = \overline{H}$. By [3], for any $x \in W$ there exists a unique self-dual element $\underline{H}_x \in \mathcal{H}$ such that

$$\underline{H}_x \in H_x + \sum_{y<x} v\mathbb{Z}[v] H_y.$$

This implies that $\{\underline{H}_x\}_{x \in W}$ is a basis for $\mathcal{H}$, the Kazhdan–Lusztig basis.

For any $x, y \in W$ the Kazhdan–Lusztig polynomials $h_{y,x} \in \mathcal{L}$, originally introduced in [3], are defined by the equality

$$\underline{H}_x = \sum_{y \leq x} h_{y,x} H_y.$$

Let $W_f = \langle S_f \rangle \subset W$ be a parabolic subgroup of W with $S_f \subset S$. One can consider the Hecke algebra corresponding to (W_f, S_f), that is $\mathcal{H}_f = \mathcal{H}(W_f, S_f)$. If we fix $u \in \{-v, v^{-1}\}$ we have that associating $H_s \mapsto u$ gives a surjection of $\mathcal{L}$-algebras $\mathcal{H}_f \to \mathcal{L}$ so that $\mathcal{L} = \mathcal{L}(u)$ becomes an $\mathcal{H}_f$-bimodule. Let W^f be the set of minimal length representatives for the right cosets $W_f \setminus W$. The two right $\mathcal{H}$-modules

$$\mathcal{M} = \mathcal{L}(v^{-1}) \otimes_{\mathcal{H}_f} \mathcal{H} \quad \text{and} \quad \mathcal{N} = \mathcal{L}(-v) \otimes_{\mathcal{H}_f} \mathcal{H}$$

are such that $M_x = 1 \otimes H_x$ and $N_x = 1 \otimes H_x$ with $x \in W^f$ are $\mathcal{L}$-basis. One can consider the induced involution on $\mathcal{M}$ (similarly on $\mathcal{N}$) and, by [1], one has that for any $x \in W^f$ there exists a unique self-dual element $\underline{M}_x \in \mathcal{M}$ (resp. $\underline{N}_x \in \mathcal{N}$) such that $\underline{M}_x \in M_x + \sum_y v\mathbb{Z}[v] M_y$ (resp. $\underline{N}_x \in N_x + \sum_y v\mathbb{Z}[v] N_y$). This allows one to introduce the parabolic Kazhdan–Lusztig polynomials $m_{y,x}$ from the equality $\underline{M}_x = \sum_{y \leq x} m_{y,x} M_y$ (resp. $n_{y,x}$ from the equality $\underline{N}_x = \sum_{y \leq x} n_{y,x} N_y$).

Kazhdan–Lusztig polynomials and the Kazhdan–Lusztig basis satisfy several positivity properties: the former are such that $h_{y,x}(v) \in \mathbb{Z}_{\geq 0}[v]$ while, if we write $\underline{H}_x \underline{H}_y = \sum_z \mu_{x,y}^z \underline{H}_z$, we have that $\mu_{x,y}^z \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$. These properties were proved for Weyl groups in [4] by using intersection cohomology of Schubert varieties and in [2] for arbitrary Coxeter systems using Soergel bimodules.

Kazhdan–Lusztig polynomials evaluated at one turn out to be closely related to some multiplicities: inversion formulae for Kazhdan–Lusztig polynomials and

BGG-reciprocity allow one to state the Kazhdan–Lusztig conjecture in the principal block of the category $\mathcal{O}$ as

$$(P_{y\cdot\mu} : \Delta_{x\cdot\mu}) = [\Delta_{x\cdot\mu} : L_{y\cdot\mu}] = h_{y_0x,y_0y}(1),$$

where y_0 is the longest element.

Finally, translation functors can be used in order to compare and control information between blocks of the category $\mathcal{O}$.

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Reductive Groups I: Irreducible Representations of Reductive Groups

Daniel Le

The goal of this lecture is to classify irreducible representations of reductive groups and to introduce the question of finding character formulas. Some references for this material include [1, Lecture II] and [2, II.1-II.5]. Throughout, we fix an algebraically closed field k. We will use the term *group* for a connected linear algebraic group over k. A subgroup of a group will refer to a closed k-subgroup scheme. A group is called *unipotent* if it is isomorphic to a subgroup of the group of unipotent upper triangular matrices of GL_n for some n. An example is the additive group $\mathbb{G}_a$. A group is called *reductive* if it has no nontrivial normal unipotent subgroups. Examples include (products of) GL_n, SO_n, and Sp_{2n}. Products of $\mathbb{G}_m = \mathrm{GL}_1$ are called *tori*.

To a reductive group G, one can associate a *root datum* as follows (see [1, §4]). To a maximal torus T (which is unique up to conjugation), one associates the character and cocharacter groups X and $X^\vee$, respectively. The Lie algebra of G decomposes into eigenspaces for T-characters, of which the nontrivial ones are called *roots* and comprise R. To each root, one can associate a corresponding *coroot* in $X^\vee$ using, for instance, the Jacobson–Morozov theorem. The set of coroots is denoted $R^\vee$. The quadruple $(R \subset X, R^\vee \subset X^\vee)$ satisfy the properties which define the notion of a root datum:

- X and $X^\vee$ are finitely generated free abelian groups;
- there is a natural perfect pairing $\langle \cdot, \cdot \rangle : X \times X^\vee \to \mathbb{Z}$;
- there is a bijection $\cdot^\vee : R \to R^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in R$;

- for any $\alpha \in R$, the involution
$$\begin{aligned} s_\alpha : X &\to X \\ \lambda &\mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \end{aligned}$$
stabilizes R; and
- for any $\alpha \in R$, the induced action of s_α on $X^\vee$ by duality stabilizes $R^\vee$.

A theorem of Chevalley states that the above process defines a bijection between reductive groups up to isomorphism and root data up to isomorphism.

Let B^+ be a *Borel subgroup*, i.e. a maximal solvable subgroup, of a reductive group G. For example, the subgroup of upper triangular matrices is a Borel subgroup of GL_n. We let R^+ be the set of roots appearing in the T-decomposition of the Lie algebra of B^+ and call these roots *positive*. Then we define the set X_+ of *dominant* characters to be
$$\{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \, \forall \, \alpha \in R^+\}.$$
We let the set R^- of *negative* roots be the set of inverses of the positive roots. Then $R = R^+ \coprod R^-$. Let B be the *opposite Borel subgroup* characterized by the property that the roots appearing in the decomposition of the Lie algebra of B are precisely the negative roots. In the above example, B is the subgroup of lower triangular matrices. Let U^+ and U be the maximal unipotent subgroups of B^+ and B, respectively.

The significance of Borel subgroups comes from a theorem of Borel that the action of a solvable group on a proper k-scheme has a fixed point. This follows from the facts that every group action has a closed orbit (by passing to orbit boundaries) and that proper quotients of solvable groups are 0-dimensional. The following important consequences are obtained by considering group actions on projectivizations of representations.

(1) Every nonzero B-representation has a one-dimensional subrepresentation on which U acts trivially.
(2) Every G-representation admits a nonzero G-equivariant map to the global sections of a G-equivariant line bundle on G/B.

Now G-equivariant line bundles are classified by T-characters as follows. For $\lambda \in X$, let $\mathcal{O}(\lambda)$ be the G-equivariant invertible sheaf on G/B with sections
$$\mathcal{O}(\lambda)(\mathcal{V}B/B) = \{f : \mathcal{V}TU/U \to k \mid f(gtU) = \lambda(t^{-1})f(gU) \, \forall \, g \in \mathcal{V}, t \in T\}$$
for any open set $\mathcal{V} \subset G$. When $G = \mathrm{SL}_2$, the space $\Gamma(\mathcal{O}(\lambda))$ of global sections is naturally identified with the space of homogeneous polynomials in two variables of a fixed degree (depending on λ). In general, one can use the fiber bundle $G/B \to G/P_\alpha$ where α is a simple root to show that $\Gamma(\mathcal{O}(\lambda)) \neq 0$ if and only if $\lambda \in X_+$ by reducing to the SL_2-case. When $\lambda \in X_+$, we write ∇_λ for $\Gamma(\mathcal{O}(\lambda))$.

When $\operatorname{char} k \neq 0$, ∇_λ may not be irreducible. For example, if $\operatorname{char} k = p$, then $\langle X^p, Y^p \rangle$ is an SL_2-invariant subspace of the space of degree p polynomials in X and Y. Nevertheless, the density of U^+B/B in G/B implies that $\nabla_\lambda^{U^+}$ is one-dimensional so that ∇_λ necessarily has irreducible socle, which we denote L_λ. Then the previous discussion implies a theorem of Chevalley: $\lambda \mapsto L_\lambda$ defines a bijection between X_+ and the set of irreducible G-representations up to isomorphism.

It is natural to ask for a formula for the dimension or even the *character* of the irreducibles L_λ. This is not easy and is a central question of this workshop. On the other hand, there is the Weyl character formula for the characters of the representations ∇_λ. This formula is a consequence of projection formulas for sheaf pushforwards and Kempf's vanishing theorem which states that the higher cohomology of $\mathcal{O}(\lambda)$ vanishes when $\lambda \in X_+$ (see [2, II.5]). In positive characteristic, this vanishing theorem can also be proved using projection formulas and using the special relationship between so-called Steinberg modules (certain ∇_λ) and Frobenius kernels (see [2, II.4]). In characteristic zero, Kempf vanishing follows from Kodaira vanishing or can be reduced to the positive characteristic case.

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Reductive Groups II: Borel-Weil-Bott, Linkage, Translation

David Schwein

In this talk we discussed three fundamental results in the algebraic representation theory of reductive groups G, following sections II.5, II.6, and II.7 of Jantzen's book [1]. Assume the (algebraically closed) base field of G has positive characteristic p.

The first result, the Borel-Weil-Bott theorem, concerns the line bundles $\mathcal{O}(\lambda)$ on the generalized flag variety G/B, which are constructed from a character λ of a fixed maximal torus T. The theorem computes for any Weyl-group element w the sheaf cohomology

$$\mathrm{H}^i\big(G/B, \mathcal{O}(w \bullet \lambda)\big)$$

provided that λ is in the $-\rho$-shifted fundamental alcove; here $\bullet$ denotes the dot action. When λ is not dominant, the cohomology vanishes for all i. When λ is dominant, the cohomology is supported in degree the length of w, where it is isomorphic to $\mathrm{H}^0\big(G/B, \mathcal{O}(\lambda)\big)$. In characteristic zero the theorem describes all simple modules, but in positive characteristic it describes only a small portion of them.

The second result, the linkage principle, gives a rough description of the blocks in the category $\mathrm{Rep}(G)$ of finite-dimensional algebraic representations of G. The principle in one of its forms states that there are no non-split extensions between the simple modules $L(\lambda)$ and $L(\mu)$, that is,

$$\mathrm{Ext}^1\big(L(\lambda), L(\mu)\big) = 0,$$

whenever $W_{\mathrm{aff}} \bullet_p \lambda \neq W_{\mathrm{aff}} \bullet_p \mu$, where $\bullet_p$ denotes the p-dilated dot action of the affine Weyl group W_{aff}. The principle implies that the Serre subcategory $\mathrm{Rep}_\lambda(G)$ generated by the simple modules $\big\{L(w \bullet_p \lambda) \mid w \in W_{\mathrm{aff}}\big\}$ is a summand of $\mathrm{Rep}(G)$.

At the end of the talk we proved the linkage principle for large primes by analyzing the infinitesimal and algebraic central characters of simple modules.

The third result is more properly described as a collection of properties of certain functors

$$T_\lambda^\mu\colon \operatorname{Rep}_\lambda(G) \to \operatorname{Rep}_\mu(G),$$

called the translation functors and defined by the formula

$$T_\lambda^\mu(V) = \operatorname{pr}_\mu\big(L(\nu) \otimes V\big).$$

Here $\operatorname{pr}\colon \operatorname{Rep}(G) \to \operatorname{Rep}_\lambda(G)$ is projection and ν is the unique dominant element in the Weyl orbit of $\mu - \lambda$. The functor T_λ^μ is an equivalence of categories whenever μ and λ lie in the same facet under the p-dilated dot action. In general, information propagates from larger to smaller facets in the closure ordering: such information includes higher induced modules (that is, the cohomologies discussed above), simple modules, and character formulas.

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Perverse Sheaves on Flag Varieties

Jens Niklas Eberhardt

1. Perverse sheaves

Perverse sheaves are a generalization of local systems. They arise naturally as derived solutions of certain linear PDEs with regular singularities by the *Riemann–Hilbert correspondence*, see [11, 15]. Moreover, they provide a home for Goresky–MacPherson's *intersection cohomology* for singular spaces, see [10]. The standard definition works via a *perverse t-structure* on the constructible derived category of sheaves, see [2], a source we will closely follow.

1.1. **Assumptions.** Let k be a field and X a complex algebraic variety, equipped with the analytic topology, together with a finite Whitney stratification $X = \biguplus_{\lambda\in\Lambda} X_\lambda$ into smooth connected strata X_λ of dimension $d_\lambda = \dim_{\mathbb{C}}(X_\lambda)$ such that $\iota_\lambda : X_\lambda \to X$ is locally closed and affine.

1.2. **Local Systems.** The category of *local systems* $\operatorname{Loc}(X_\lambda, k)$ is the full subcategory of locally constant sheaves on X_λ with finitely generated stalks. In order to obtain a category closed with respect to Verdier duality is convenient to consider $\operatorname{Loc}(X_\lambda, k)[d_\lambda] = D^b_{lc}(X,k)^{t=-d_\lambda}$ the heart of the standard t-structure—shifted by d_λ—of $D^b_{lc}(X,k)$, the subcategory of the bounded derived category of sheaves on X_λ whose cohomology sheaves are local systems.

1.3. **Perverse Sheaves.** To obtain an Abelian category of sheaves on X that is stable under Verdier duality, one glues the shifted standard t-structures on the strata to a so-called *perverse t-structure* on the Λ-constructible bounded derived category of sheaves $\mathrm{D}^b_\Lambda(X,k)$ which consists of sheaves that restrict to $\mathrm{D}^b_{lc}(X_\lambda,k)$ for each $\lambda\in\Lambda$. The category of Λ-constructible *perverse sheaves* on X is the heart of the perverse t-structure $\mathrm{Perv}_\Lambda(X,k)=\mathrm{D}^b_\Lambda(X,k)^{t=0}$ and a complex of sheaves $\mathcal{F}$ is in $\mathrm{Perv}_\Lambda(X,k)$ if and only if $\mathcal{H}^i(\iota_\lambda^*(\mathcal{F}))=\mathcal{H}^j(\iota_\lambda^!(\mathcal{F}))=0$ for all $\lambda\in\Lambda$ and $i>-d_\lambda>j$.

1.4. **Intersection Cohomology.** An important example of perverse sheaves are the *intersection cohomology complexes* $\mathrm{IC}(\overline{X}_\lambda,\mathcal{L})$ that arise as minimal extensions of local systems $\mathcal{L}\in\mathrm{Loc}(X_\lambda,k)$. The cohomology of $\mathrm{IC}(\overline{X}_\lambda,k)$ is the intersection cohomology of $\overline{X}_\lambda$ and there is a bijection

$$\{(\lambda,\mathcal{L})\mid\mathcal{L}\in\mathrm{Irr}(\mathrm{Loc}(X_\lambda))\}\xrightarrow{\sim}\mathrm{Irr}(\mathrm{Perv}_\Lambda(X)),(\lambda,\mathcal{L})\mapsto\mathrm{IC}(\overline{X}_\lambda,\mathcal{L}).$$

1.5. **Highest Weight Category.** If each stratum X_λ is simply-connected and $H^2(X_\lambda,k)=0$, the category $\mathrm{Perv}_\Lambda(X,k)$ has the structure of a *highest weight category* with standard and costandard objects given by $\Delta_\lambda=\iota_{\lambda,!}(k[d_\lambda])$ and $\nabla_\lambda=\iota_{\lambda,*}(k[d_\lambda])$, respectively. This implies nice homological properties such as the existence of projective, injective and tilting objects in $\mathrm{Perv}_\Lambda(X,k)$, see [1].

1.6. **An Example.** Let $X=\mathbb{C}=\{0\}\uplus\mathbb{C}^\times$. A perverse sheaf $\mathcal{F}\in\mathrm{Perv}_\Lambda(\mathbb{C},k)$ is completely determined by the (relative) cohomology groups $\mathrm{can}:H^{-1}(\{1\};\mathcal{F})\leftrightarrows H^0(\mathbb{C},\{1\};\mathcal{F}):\mathrm{var}$ together with a *canonical* and *variational morphism* satisfying $\mathrm{var}\circ\mathrm{can}=h-1$ for the monodromy action of the clockwise loop $h\in\pi_1(\mathbb{C}^\times,1)$. By adding a point at infinity, there is a similar description for the category of perverse sheaves on $X=\mathbb{P}^1_\mathbb{C}$ stratified by a point and a line where the monodromy h becomes trivial. Hence, there is an equivalence between perverse sheaves and representations of a quiver $\mathrm{Perv}_\Lambda(\mathbb{P}^1_\mathbb{C},k)\xrightarrow{\sim}\{(\mathrm{can}:\psi\leftrightarrows\varphi:\mathrm{var})\in k\text{-mod}\mid\mathrm{var}\circ\mathrm{can}=0\}$. The principal block of the BGG category $\mathcal{O}$ of highest weight representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has the same description, by mapping a representation to its two highest weight spaces with the action of the raising and lowering operator—a first hint for a deep connection between perverse sheaves and representation theory.

1.7. **Further topics.** The *decomposition theorem* shows the semi-simplicity of pushforwards of intersection cohomology complexes along proper algebraic maps if $\mathrm{char}(k)=0$, see [2, 16, 6, 7], and there is a formalism of *nearby/vanishing cycles* and *gluing* of perverse sheaves, widely generalizing Example 1.6, see [1].

2. Flag varieties

Perverse sheaves on flag varieties are closely related to the BGG category $\mathcal{O}$ of complex reductive Lie algebras and are a crucial ingredient in the proof of the Kazhdan–Lusztig conjectures for characters of simple highest weight modules.

2.1. **Bruhat stratification.** Denote by $G \supset B \supset T$ a complex reductive group together with a Borel subgroup and maximal torus. Then, denoting by $W = \mathrm{N}_G(T)/T$ the Weyl group, the flag variety $X = G/B$ has a natural stratification (B) into its B-orbits $X = \biguplus_{w \in W} X_w$, where $X_w = BwB/B \cong \mathbb{C}^{\ell(w)}$, and one can hence consider the category of perverse sheaves $\mathrm{Perv}_{(B)}(X, k)$.

2.2. **Kazhdan–Lusztig conjectures.** Grothendieck's function sheaf correspondence suggests that sheaves $\mathcal{F} \in \mathrm{D}^b_{(B)}(X, k)$ are closely related to the Hecke algebra $\mathcal{H}$ associated to W via

$$h(\mathcal{F}) = \sum_{w \in W} \sum_{i \in \mathbb{Z}} \dim_k H^{\ell(v)-i}(\iota_w^* \mathcal{F}) v^i h_w \in \mathcal{H} = \bigoplus_{w \in W} \mathbb{Z}[v, v^{-1}] h_w$$

where we use Soergel's conventions for the Hecke algebra, see [18]. Pursuing the philosophy further, interesting sheaves on X yield interesting elements in the Hecke algebra.

Theorem 1 (Kazhdan–Lusztig [14, 13])**.** *Let* $\mathrm{char}(k) = 0$. *For* $w \in W$ *we have that* $h(\mathrm{IC}(\overline{X}_w, k)) = \underline{h}_w$ *where* $\underline{h}_w \in \mathcal{H}$ *denotes the Kazhdan–Lusztig basis element.*

This theorem, combined with the Riemann–Hilbert correspondence [11, 15] and the localisation theorem [3, 12] provides an algorithm to compute the characters of simple highest weight modules of complex reductive Lie algebras, proving a conjecture of Kazhdan–Lusztig.

2.3. **Further results.** There is a deep symmetry, known as *Koszul duality* between so-called *mixed* perverse sheaves (see [19, 8, 9]) of Langlands dual flag varieties, which yields a different proof of the Kazhdan–Lusztig conjectures, see [17, 4]. Many of these results extend to the setting of flag varieties of Kac–Moody groups, such as affine Grassmannians, see for example [5].

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The Hecke Algebra and Hecke Category

Patrick Bieker

We explain how to categorify the Hecke algebra geometrically following [1, §2].

1. The Hecke algebra of a reductive group

Let G be a split reductive group over a finite field $\mathbb{F}_q$ and let $T \subset B \subset G$ be a split maximal torus and a Borel subgroup, respectively. The Hecke algebra of G is defined as

$$\mathcal{H}(G,B) = \mathrm{Fun}_{B(\mathbb{F}_q)\times B(\mathbb{F}_q)}(G(\mathbb{F}_q),k)$$

the set of $B(\mathbb{F}_q)$-biinvariant functions on $G(\mathbb{F}_q)$ with values in k, which we for now assume to be an algebraically closed field in characteristic 0. The algebra structure on $\mathcal{H}(G,B)$ is given by convolution, in other words, for $f, f' \in \mathcal{H}(G,B)$ we define

$$(f * f')(g) = \frac{1}{|B(\mathbb{F}_q)|}\sum_{h\in G(\mathbb{F}_q)} f(gh^{-1})f'(h).$$

The algebra $\mathcal{H}(G,B)$ is associative and unital (the normalisation in the convolution is made in such a way that the constant function on $B(\mathbb{F}_q)$ is the neutral element).

By the Bruhat decomposition $G(\mathbb{F}_q) = \coprod_{w\in W} B(\mathbb{F}_q)\cdot wB(\mathbb{F}_q)$, the $B(\mathbb{F}_q)$-double cosets are enumerated by the Weyl group W. In particular, the Hecke algebra $\mathcal{H}$ has a basis $\{t_w\}_{w\in W}$ given by indicator functions of the double cosets $B(\mathbb{F}_q)\cdot wB(\mathbb{F}_q)$ of $G(\mathbb{F}_q)$.

To G (and the choice of T and B) we can associate its Coxeter system (W, S). In a previous talk, the Hecke algebra $\mathcal{H}(W, S)$ associated to a Coxeter system was defined as the free $\mathcal{L} = \mathbb{Z}[v^{\pm 1}]$-module

$$\bigoplus_{x \in W} \mathcal{L} T_x$$

with algebra structure defined by $T_x T_y = T_{xy}$ for $x, y \in W$ with $\ell(xy) = \ell(x) + \ell(y)$ and $T_s^2 = v^{-2} T_e + (v^{-2} - 1) T_s$ for $s \in S$. In order to compare the two notions of Hecke algebras we regard k as an algebra over $\mathbb{Z}[v^{\pm 1}]$ via $v \mapsto q^{-\frac{1}{2}}$.

Lemma 1. *The map*

$$\begin{aligned} \mathcal{H}(W, S) \otimes_{\mathcal{L}} k &\to \mathcal{H}(G, B) \\ T_x &\mapsto t_x \end{aligned}$$

induces a well-defined isomorphism of k-algebras.

Grothendieck's function-sheaf correspondence tells us how to categorify the construction of the Hecke algebra: Instead of considering (equivariant) functions on G we should consider (equivariant) sheaves on G.

2. (Geometric) Categorification of the Hecke algebra

Let now k denote an arbitrary field. A sheaf on a variety X over $\mathbb{C}$ will without further mention always mean a sheaf of k-vector spaces with respect to the classical topology on $X(\mathbb{C})$. As we will be mainly interested in groups over the complex numbers, we shift our perspective: We fix a Kac-Moody root datum and we denote by $G \supset B \supset T$ now the associated Kac-Moody group (over $\mathbb{C}$) with corresponding Borel B and maximal torus T. The main examples of Kac-Moody groups of interest for us are reductive groups and (central extensions of) loop groups. In general, G is an ind-variety over $\mathbb{C}$. We denote by G/B the corresponding flag variety, which in general is an ind-projective ind-variety. The Bruhat decomposition induces the decomposition $G/B = \coprod_{w \in W} X_w$ of the flag variety into its Schubert cells $X_w = BwB/B$.

We denote by $D^b_B(G/B, k)$ the bounded derived category of k-vector spaces on the flag variety following [2] and [3]. In particular, we require elements of $D^b_B(G/B, k)$ to be supported on only finitely many Schubert cells. As the action of B on G is free, the category $D^b_B(G/B, k)$ serves as an analogue of $B \times B$ equivariant functions on G.

As a next step, we define a convolution on $D^b_B(G/B, k)$ categorifying the convolution of functions. The idea (which can also be made precise in our setting) is the following. We consider equivariant sheaves on G/B as sheaves on the (stacky) double quotient $X = B \backslash G / B$. We get the convolution diagram

$$X \times X \xleftarrow{(p_1, p_2)} B \backslash G \times^B G/B \xrightarrow{m} X,$$

where the right map is given by the multiplication map on G. Then convolution on $D^b_B(G/B, k)$ is defined as

$$\mathcal{F} \star \mathcal{G} = m_*(p_1^* \mathcal{F} \otimes p_2^* \mathcal{G}).$$

It remains to construct analogues of the generators of the Hecke algebra inside $D^b_B(G/B,k)$. Namely, for a simple reflection $s \in S$ we have the parabolic subgroup $P_s = \overline{BsB} = BsB \cup B \subset G$. We denote by $k_{P_s/B}$ the corresponding constant sheaf on the flag variety (this is clearly B-equivariant).

Definition 2. The *Hecke category* (in its geometric incarnation) is defined as

$$\mathcal{H}^k_{\text{geom}} = \langle k_{P_s/B} \colon s \in S \rangle_{*,\oplus,[1],\text{Kar}}$$

the full subcategory of $D^b_B(G/B,k)$ generated by $k_{P_s/B}$ under convolution, direct sums, shifts and direct summands.

Note that both the equivariant derived category $D^b_B(G/B)$ as well as the Hecke category $\mathcal{H}^k_{\text{geom}}$ are *Krull-Schmidt categories.* The indecomposable objects in $\mathcal{H}^k_{\text{geom}}$ are given by the indecomposable parity sheaves.

In order to compare the Hecke category with the Hecke algebra, we denote by $[\mathcal{H}^k_{\text{geom}}]_\oplus$ the split Grothendieck group of $\mathcal{H}^k_{\text{geom}}$. $[\mathcal{H}^k_{\text{geom}}]_\oplus$ is a $\mathbb{Z}[v^\pm]$-algebra via convolution $[\mathcal{F}] \cdot [\mathcal{G}] = [\mathcal{F} * \mathcal{G}]$ and $v \cdot [\mathcal{F}] = [\mathcal{F}[1]]$. We denote the Kazhdan-Lusztig basis of $\mathcal{H}(G,B)$ by $\underline{H}_w$ for $w \in W$ and the standard basis by H_w for $w \in W$. Then $\mathcal{H}^k_{\text{geom}}$ categorifies the Hecke algebra in the following sense.

Theorem 3. *The map*

$$\begin{aligned} \mathcal{H}(G,B) &\to [\mathcal{H}^k_{geom}]_\oplus \\ \underline{H}_s &\mapsto [k_{P_s/B}[1]] \end{aligned}$$

for $s \in S$ defines an isomorphism of $\mathbb{Z}[v^\pm]$-algebras. The inverse isomorphism is given by the character map

$$\begin{aligned} \text{ch} \colon [\mathcal{H}^k_{geom}]_\oplus &\to \mathcal{H}(G,B) \\ \mathcal{F} &\mapsto \sum_{w \in W} \sum_{i \in \mathbb{Z}} \dim(H^i(\mathcal{F}_{xB/B})) v^{-\ell(w)-i} H_w, \end{aligned}$$

where $\mathcal{F}_{xB/B}$ denotes the stalk of $\mathcal{F}$ at the point wB/B of G/B.

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Soergel Bimodules

Arthur Garnier

The category of Soergel bimodules is an algebraic generalization of the geometric Hecke category. More precisely, if G is a connected reductive algebraic group over $\mathbb{C}$, with a Borel subgroup $B < G$ containing a maximal torus T, with associated Weyl group W and if we let $R := \mathrm{Sym}(X^*(T) \otimes \mathbb{Q})$ (with $\deg(X^*(T)) = 2$), then we have the hypercohomology functor

$$D^b_B(G/B, \mathbb{Q}) \xrightarrow{\mathbb{H}^*_B} H^*_B(G/B, \mathbb{Q})\text{−}\mathbf{gmod} = (R \otimes_{R^W} R)\text{−}\mathbf{gmod} \xrightarrow{\text{proj.}} R\text{−}\mathbf{gbim},$$

where R−**gbim** is the category of graded R-bimodules. By a theorem of Soergel, the restriction of this functor to the *geometric Hecke category* (the category of semisimple perverse sheaves on G/B) is fully faithful. Its image in R−**gbim** is the category of Soergel bimodules.

This still makes sense for any Coxeter system (W, S), with a sufficiently nice faithful reflection representation $\mathfrak{h}$ of W. In this setting, we introduce the category of Bott-Samelson bimodules and a Soergel bimodule is then defined to be a direct summand of a finite direct sum of shifts of Bott-Samelson bimodules. Such bimodules form an additive monoidal category denoted by SBim.

We review the basic properties of SBim, in particular the Δ-filtrations and the *character*. By Soergel's categorification theorem, the character is an isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras

$$\begin{array}{cccccc} \mathrm{ch} & : & [\mathrm{SBim}]_\oplus & \xrightarrow{\sim} & H(W,S) \\ & & [B] & \longmapsto & \mathrm{ch}(B) \end{array}$$

where $H(W, S)$ is the Hecke algebra of (W, S). Then, we state the classification of indecomposable bimodules, which are parametrized by W and we denote by B_w the indecomposable bimodule associated to $w \in W$.

Soergel's conjecture states that we have $\mathrm{ch}(B_w) \in H(W, S)$ is the element of the Kazhdan-Lusztig basis of $H(W, S)$ corresponding to $w \in W$. In the last part, we discuss the relation between this conjecture (now a theorem of Elias and Williamson) and Lusztig's multiplicity conjecture. First, we notice that Soergel's conjecture also implies the Kazhdan-Lusztig positivity conjecture (stating that the Kazhdan-Lusztig polynomials all have non-negative coefficients). Secondly, we introduce the Soergel modules and Soergel's functor $\mathbb{V}$. We review the main features of this functor and in particular, we state a theorem of Soergel stating that $\mathbb{V}$ establishes an equivalence between the category of projective objects in the principal block $\mathcal{O}_0$ of the category $\mathcal{O}$ and the category of ungraded Soergel modules. We finish by deducing Lusztig's conjecture from Soergel's conjecture for the geometric representation of the Weyl group W.

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Parity Sheaves

Eva Viehmann

Parity sheaves are a class of constructible complexes on certain stratified varieties that are defined by some parity vanishing condition on their cohomology. They are introduced in work of Juteau, Mautner and Williamson and have rich applications in geometric representation theory. In cases where the sheaf coefficients are of characteristic 0, they coincide with IC sheaves and are otherwise a useful replacement.

Let X be a variety X over $\mathbb{C}$ together with a stratification $X = \coprod_{\lambda\in\Lambda} X_\lambda$. We consider coefficients in a complete local principal ideal domain k.

Assume that $f : Y \to X$ is a stratified map with Y smooth. Then we would like to understand the indecomposable summands of $f_*\underline{k}_Y$ where $\underline{k}_Y$ is the constant sheaf on Y. In characteristic 0 this can be achieved using the decomposition theorem which asserts that $f_*\underline{k}_Y$ is a certain sum of IC sheaves associated with suitable local systems on the strata.

In general this need no longer be the case. More precisely, under the assumption that f is semi-small, [1] prove that the decomposition theorem only holds for $f_*\underline{k}_Y$ if certain intersection forms associated to the strata are all non-degenerate.

The easiest example of this phenomenon is the nilpotent cone in $\mathfrak{sl}_2$, which can also be described as the variety

$$X = \{(x, y, z) \mid x^2 = -yz\} \subseteq \mathbb{C}^3.$$

Let $f : \tilde{X} \to X$ be the blow-up of the unique singular point 0. Computing the above-mentioned intersection forms we obtain that for char $k \neq 2$, we have a decomposition

$$f_*(k_{\tilde{X}})[2] \cong k_X[2] \oplus k_{\{0\}}.$$

In characteristic 2, however, $f_*(k_{\tilde{X}})[2]$ turns out to be indecomposable, and is a first example of a parity sheaf.

Parity sheaves are defined abstractly as being certain indecomposable elements in the bounded Λ-constructible derived category of k-sheaves on X satisfying a parity condition on their cohomology and a condition to be shifted to a specific degree. Juteau, Mautner and Williamson show that parity sheaves are uniquely determined by their support, which agrees with the closure of a single stratum X_λ, and their restriction to X_λ, which is a local system of free finite rank k-modules on X_λ.

A main result on parity sheaves is that they satisfy a direct analog of the decomposition theorem in the sense that $f_*(k_Y)$ can be decomposed as a direct sum of shifts of parity sheaves, with multiplicities that are again computable using intersection forms on the various strata.

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The Diagrammatic Hecke Category

Leonardo Maltoni

Recall that, given a Coxeter system (W, S) with a realization $\mathfrak{h}$ over a ring k, the Hecke category is a certain graded k-linear monoidal category whose Grothendieck ring is the Hecke algebra of (W, S). It has several incarnations (for instance Soergel bimodules, or parity sheaves) and it can be given a presentation by generators and relations using the language of diagrams for strict monoidal categories.

More generally, strict 2-categories can be described diagrammatically: a 2-morphism will be represented by a planar diagram, where generic points correspond to objects and generic horizontal lines correspond to 1-morphisms. The axioms of strict 2-categories can then be restated by declaring that two diagrams represent the same 2-morphism if they are equivalent up to rectilinear isotopy. If, furthermore, all 1-morphims have bi-adjoints and all 2-morphisms are cyclic, then one can extend the equivalence to usual isotopy. (Strict) monoidal categories can be viewed as (strict) 2-categories with one object, by seeing the objects of the monoidal category as 1-morphisms and the morphisms as 2-morphims. One can then hope to obtain such a diagrammatic description for the Hecke category.

In [3], Elias and Williamson, based on previous work of Elias [1] and Elias-Khovanov [2], introduced a category whose objects are sequences of colored points (one color for each simple reflection in S) and morphisms are linear combinations of equivalence classes of certain planar colored graphs with boundary, contained in the strip $\mathbb{R} \times [0, 1]$. The boundary points in $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ correspond to the source and the target respectively. More precisely, these graphs are obtained by composition from a specific list of generators and are identified up to isotopy and by several other relations.

Elias and Williamson also proved that, under the hypotheses of Soergel's categorification theorem, this category is equivalent to the category of Soergel bimodules, but the diagrammatic versions of the categorification statement and of the classification of the indecomposable objects hold under milder assumption on the realization $\mathfrak{h}$, that can in particular be easily satisfied also in the case of an affine Weyl group in positive characteristic. Furthermore, Riche and Williamson proved in [5] that, in the crystallographic case, the diagrammatic category is equivalent to the category of parity sheaves over the (affine) flag variety.

The indecomposable objects define a very interesting basis of the Hecke algebra, called the p-canonical basis. Its computation is more complicated than the classical Kazhdan-Lusztig basis. For example one can obtain it by computing the graded rank of some intersection forms either in the geometric or in the diagrammatic setting. Many examples can be found in [4].

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Lusztig's Conjecture

Matthew Westaway

Let G be a connected reductive algebraic group over an algebraically closed field $\mathbf{K}$ of characteristic $p \geq 0$. One of the most fundamental problems in the representation theory of algebraic groups is to determine the characters of the irreducible G-modules. Recall that the character of a (finite-dimensional) G-module M is defined by

$$\mathrm{ch}(M) := \sum_{\lambda \in X(T)} (\dim M_\lambda) e^\lambda \in \mathbf{Z}[X(T)],$$

where $X(T)$ is the character group of a maximal torus T of G and M_λ denotes the λ-weight space of M. Characters encode lots of information about their respective modules, and so their calculation is highly desirable.

When the characteristic of $\mathbf{K}$ is zero, the irreducible modules are precisely the Weyl modules (denoted $\Delta(\lambda)$, and indexed by the dominant weights $\lambda \in X(T)^+$) and their characters are given by Weyl's character formula. For $p > 0$, however, Weyl modules are generally not irreducible; instead, each irreducible module can be found as the head of a (unique) Weyl module (we denote such heads $L(\lambda)$). Nonetheless, it turns out that $\mathrm{ch}(\Delta(\lambda))$ is given by Weyl's character formula in all characteristics, and so the characters of the irreducible modules $L(\lambda)$ may be calculated by determining the integers $m_{\lambda,\mu}$ in the expression

$$[L(\lambda)] = \sum_{\mu \in X(T)^+} m_{\lambda,\mu} [\Delta(\mu)]$$

in the Grothendieck group of $\mathrm{Rep}(G)$ (such an expression is possible, as the Weyl modules give a basis of said Grothendieck group). Via the application of translation functors (see, for example, [9]), this computation may be reduced to the setting of the *principal block*: letting W_{aff} denote the affine Weyl group of (G,T) and $\cdot_p$ denote the p-dilated dot-action of W_{aff} on $X(T)$, this means that it is sufficient to determine (for appropriate $w \in W_{\mathrm{aff}}$) the integers $m_{x,w}$ such that

$$[L(w \cdot_p 0)] = \sum_{\substack{x \in W_{\mathrm{aff}} \\ x \cdot_p 0 \in X(T)^+}} m_{x,w}[\Delta(x \cdot_p 0)]$$

in the Grothendieck group. It is in this setting where Lusztig, in [16], made his eponymous conjecture (which Jantzen had already shown to hold in types A_1, A_2, A_3, B_2 and G_2, see [8, 9]):

Conjecture (Lusztig's Conjecture [16]). Let h be the Coxeter number of (G,T), and assume $p \geq h$. Suppose further that $w \in W_{\mathrm{aff}}$ is such that $w \cdot_p 0 \in X(T)^+$ and

$$\langle w \cdot_p 0 + \rho, \alpha^\vee \rangle \leq p(p-h+2)$$

for all positive roots α. Then

$$[L(w \cdot_p 0)] = \sum_{\substack{x \leq w \\ x \cdot_p 0 \in X(T)^+}} (-1)^{l(w)+l(x)} h_{w_0 x, w_0 w}(1)[\Delta(x \cdot_p 0)],$$

where $h_{a,b} \in \mathbf{Z}[v, v^{-1}]$ denotes the affine Kazhdan-Lusztig polynomial for $a, b \in W_{\mathrm{aff}}$, and w_0 is the longest element in the Weyl group of (G,T).

The (slightly awkward) requirement that $\langle w \cdot_p 0 + \rho, \alpha^\vee \rangle \leq p(p-h+2)$ for all positive roots α is known as Jantzen's condition, and it has long been known that Lusztig's conjecture can fail without that assumption. Nonetheless, Kato proved in [10] that Lusztig's conjecture is compatible with the Steinberg tensor product theorem, which reduces understanding the irreducible modules to understanding those with highest weight in

$$X_1 := \{\lambda \in X(T) \mid 0 \leq \langle \lambda, \alpha^\vee \rangle < p \text{ for all simple } \alpha\}.$$

It was therefore deemed reasonable to replace Jantzen's condition with the requirement that $w \cdot_p 0 \in X_1$, which, through the Steinberg tensor product theorem, would mean that Lusztig's conjecture determines the characters of *all* irreducible G-modules.

One remarkable observation about this conjecture is that the coefficients on the right hand side are independent of the prime p! That remarkable fact comes with a cost, however: the conjecture is not true in the generality stated above. In 2013, Williamson [24, 25] showed that Lusztig's conjecture fails for a large collection of the primes for which it was conjectured to hold.

If p is sufficiently large, however, Lusztig's conjecture is indeed true. This was proved in the early 1990s, through a scheme laid out by Lusztig [17, 18] (with appropriate modifications for the non-simply-laced cases made by Lusztig in [19, 20]). The scheme first relates Lusztig's conjecture to an analogous statement for quantum groups [2], then relates quantum groups to certain affine Lie algebras [11, 12, 13], and finally proves the appropriate results on affine Lie algebras [14, 15], using similar techniques as were used in the proof of the Kazhdan-Lusztig conjectures for finite-dimensional complex semisimple Lie algebras. The limitation of $p \gg 0$ comes out of the first step: Andersen, Jantzen and Soergel's proof in [2] has to pass through a localization $\mathbf{Z}[d^{-1}]$ of the integers for some (unknown) d, and thus only proves the desired connection for $p > d$. No explicit bound on p is given in [2], although work of Fiebig [7] has been able to provide one (which is, necessarily, very large). Since the 1990s, various other proofs of Lusztig's conjecture for large primes have been found, for example in [1, 4, 6].

Despite the failure of Lusztig's conjecture for smaller primes, the correct replacement for Lusztig's conjecture has been found (and now proved in a handful of different ways, see [3, 5, 22, 23]). Notably, the proof in [22] proves that the replacement holds for *all* primes (even primes smaller than h). The content of this replacement, however, is a lot more opaque than the content of Lusztig's conjecture, and a major question going forward is whether it is possible to understand it in a more tangible way.

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Combinatorial Origins of Geometric Satake

Shane Kelly

This talk was about how Lusztig's character formula leads to the prediction,

$$h_{t_\mu w_0, t_\lambda w_0}(1) = \dim \Delta_\lambda(\mu), \tag{1}$$

for the values of certain affine Kazhdan-Lusztig polynomials at 1. We follow the notation from [5, §3.5].

Here, $\lambda, \mu \in X$ are characters of the algebraic group associated to a fixed root datum $(X, \Phi, X^\vee, \Phi^\vee)$, we write $\Delta_\lambda(\mu)$ for the weight space of μ in the Weyl module Δ_λ associated to λ, the Kazhdan-Lusztig polynomials (of a choice of simple roots and elements x, y in the extended affine Weyl group $\mathcal{W} = W \ltimes X$) are written $h_{x,y}$, the longest element of the Weyl group is w_0 and t_μ, t_λ are the translations $\mathcal{W}$ in the affine Weyl group associated to $\mu, \lambda \in X$. Note that the action of $\mathcal{W}$ on X is via the p-dilated dot action $(x, t_\lambda) \bullet_p \mu = x((\mu + p\lambda) + \rho) - \rho$ where $p \in \mathbb{N}$ is a fixed chosen prime, and $\rho = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha$ is the half sum of our choice of positive roots.

Lusztig's derivation of (1) from his character formula is rather short (this derivation is a combination of the the one in [3, pp.29-30] and the one in [5, §3.5]):

$$
\begin{aligned}
&\sum_{\mu\in X_+} \dim\Delta_\lambda(\mu)\frac{1}{|\mathsf{Stab}_W(\mu)|}\sum_{x\in W}\mathsf{sgn}(x)\mathsf{ch}\Delta_{p\mu-\rho+x\rho} \\
=&\sum_{\mu\in X_+} \dim\Delta_\lambda(\mu)\frac{1}{|\mathsf{Stab}_W(\mu)|}\sum_{x\in W} e^{px\mu} && \begin{pmatrix}\text{Weyl's character}\\ \text{formula}\end{pmatrix} \\
=&\mathsf{ch}L_{p\lambda} && \begin{pmatrix}\text{Steinberg's tensor}\\ \text{product theorem}\end{pmatrix} \\
=&\mathsf{ch}L_{t_\lambda w_0 \bullet_p (-2\rho)} && (p\lambda = t_\lambda w_0 \bullet_p (-2\rho)) \\
=&\sum_{\substack{y'\leq t_\lambda w_0,\\ y'\bullet_p(-2\rho)\in X^+}} \mathsf{sgn}(y' t_\lambda w_0)\ h_{y',t_\lambda w_0}(1)\ \mathsf{ch}\Delta_{y'\bullet_p(-2\rho)} && \begin{pmatrix}\text{Lusztig character}\\ \text{formula}\end{pmatrix}
\end{aligned}
$$

The summand $\mu, x = id$ in the first expression and $y' = t_\mu w_0$ in the last expression are respectively of the forms

$$\dim\Delta_\lambda(\mu)\frac{1}{|\mathsf{Stab}_W(\mu)|}\mathsf{ch}\Delta_{p\mu} \qquad \text{and} \qquad \mathsf{sgn}(t_\mu w_0 t_\lambda w_0)\ h_{t_\mu w_0, t_\lambda, w_0}(1)\ \mathsf{ch}\Delta_{p\mu},$$

from which one can deduce (1) (for appropriate λ, μ, p).

The prediction was proved by Lusztig [4], and Kato [1], and also reproved by Knop [2].

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Geometric Satake and Finkelberg–Mirković

Leonardo Patimo

The Geometric Satake equivalence and the Finkelberg–Mirković conjecture are two central statements in geometric representation theory, both connecting the representation theory of a reductive group with the geometry of the affine Grassmannian of its Langlands dual. The Geometric Satake equivalence was proved by Mirković and Vilonen in 2007 [1], while the Finkelberg–Mirković conjecture is still open, although significant progress towards its proof has been made in the last years (cf. [2, 3]). Assuming the Finkelberg–Mirković conjecture, we can obtain a direct proof of Lusztig's character formula for large primes (cf. [4]).

The affine Grassmannian. We recall a few facts about the geometry of the affine Grassmannian. We refer to [5] for more details.

The affine Grassmannian of a reductive group G can be thought of as an infinite dimensional analogue of the usual Grassmannian. Let $\mathcal{O} = \mathbb{C}[[t]]$ be the ring of formal power series and let $\mathcal{K} = \mathbb{C}((t))$ be its quotient field, the field of formal Laurent series. Then the (space of $\mathbb{C}$-points of the) affine Grassmannian Gr_G is the quotient $G(\mathcal{K})/G(\mathcal{O})$. It is an ind-projective ind-scheme of ind-finite type, i.e. it can be realized as the direct limit of projective varieties.

If $G = GL_n$, the affine Grassmannian parametrizes $\mathcal{O}$-lattices in $\mathcal{K}^n$. In this case, we can have a direct look at its structure of an ind-projective ind-scheme. In fact, we have

$$\mathrm{Gr}_{GL_n} = \bigcup_{M\geq 0} \mathrm{Gr}^M_{GL_n}, \text{ where } \mathrm{Gr}^M_{GL_n} := \{\Lambda \subset \mathcal{K}^n \text{ lattice} \mid t^M\mathcal{O}^n \subset \Lambda \subset t^{-M}\mathcal{O}^n\}.$$

Moreover, $\mathrm{Gr}^M_{GL_n} \cong \{V \subset t^{-M}\mathcal{O}^n/t^M\mathcal{O}^n \mid V \text{ vector space over } \mathbb{C} \text{ with } tV \subset V\}$ is a closed subset of the Grassmannian of $\mathbb{C}$-vector spaces in $\mathbb{C}^{2nM}$. In particular, each $\mathrm{Gr}^M_{GL_n}$ is projective.

Let $T \subset B$ be a maximal torus and a Borel subgroup of G. Let X denote the cocharacter lattice of T and let $X_+ \subset X$ be the subset of dominant cocharacters. Every $\mu \in X$ defines a morphism $\mathbb{C}^* \to T$, hence a point in $T(\mathbb{C}[t, t^{-1}]) \subset T(\mathcal{K}) \subset G(\mathcal{K})$ and we denote by t^μ its image in Gr_G. The decomposition of Gr_G into $G(\mathcal{O})$-orbits, called the *Cartan decomposition*, has the following form:

$$\mathrm{Gr}_G = \bigsqcup_{\mu\in X_+} G(\mathcal{O}) \cdot t^\mu G(\mathcal{O})/G(\mathcal{O}).$$

The $G(\mathcal{O})$-orbit of t^μ, denoted Gr^μ_G, is called a *Schubert cell* and its closure $\overline{\mathrm{Gr}^\mu_G}$ is called a *Schubert variety.* Each Schubert variety is a projective variety of dimension $2\langle\rho, \mu\rangle$, where ρ is the half-sum of the positive roots. We have $\overline{\mathrm{Gr}^\mu_G} = \bigsqcup_{\lambda\leq\mu} \mathrm{Gr}^\lambda_G$, where $\lambda \leq \mu$ if $\mu - \lambda$ can be written as sum of positive coroots. Each Schubert variety can be obtained as an affine bundle over a finite-dimensional partial flag variety of G.

The connected components of Gr_G are parameterized by the fundamental group of G, i.e. by the quotient X/Q, with Q denoting the coroot lattice. Moreover, the dimensions of every Schubert variety in a given connected component have the same parity. This implies that there are no extensions between parity sheaves which are constructible with respect to the Cartan decomposition.

Let $Iw \subset G(\mathcal{K})$ denote the Iwahori subgroup of $G(\mathcal{K})$, i.e. the preimage of B under the morphism $G(\mathcal{O}) \to G(\mathbb{C})$ defined by $t \mapsto 0$. Each Schubert cell decomposes into Iw orbits as follows.

$$Gr^\mu_G = \bigsqcup_{\lambda\in W\mu} Iw \cdot t^\lambda G(\mathcal{O})/G(\mathcal{O}).$$

The Iwahori-orbit of t^μ is isomorphic to an affine space $\mathbb{C}^{\ell(\lambda)}$ of dimension

$$\ell(\lambda) := \sum_{\substack{\alpha\in\Phi_+\\ \langle\lambda,\alpha\rangle>0}} \langle\lambda,\alpha\rangle - \sum_{\substack{\alpha\in\Phi_+\\ \langle\lambda,\alpha\rangle<0}} (\langle\lambda,\alpha\rangle+1).$$

The Geometric Satake equivalence. The Geometric Satake equivalence states that the category of representations of a reductive group can be realized geometrically as perverse sheaves on the affine Grassmannian of the Langlands dual group. From the point of view of representation theory this is very interesting, because it implies that the geometry Gr_G governs the representation theory of $G^\vee$ in any characteristic, thus allowing to study representations in positive characteristic uniformly in p.

Let k be an arbitrary field. We consider the category $P_{G(\mathcal{O})}(\mathrm{Gr}_G, k)$ of $G(\mathcal{O})$-equivariant perverse sheaves on the affine Grassmannian with coefficients in k. Under the convolution product, $P_{G(\mathcal{O})}(\mathrm{Gr}_G, k)$ forms a monoidal category. In fact, since the multiplication $m : G(\mathcal{K}) \times_{G(\mathcal{O})} \mathrm{Gr}_G \to \mathrm{Gr}_G$ is a stratified semismall morphism, convolution of perverse sheaves is again perverse.

Theorem 1 (Geometric Satake equivalence, [1])**.** *There exists an equivalence of monoidal categories*

$$\mathcal{S} : (P_{G(\mathcal{O})}(\mathrm{Gr}_G, k), *) \xrightarrow{\sim} (Rep(G_k^\vee), \otimes_k))$$

where $G_k^\vee$ is the Langlands dual split reductive group of G constructed over k.

In the proof, one does not construct the functor $\mathcal{S}$ directly, but rather makes use of Tannakian reconstruction: one abstractly shows that the category $P_{G(\mathcal{O})}(\mathrm{Gr}_G, k)$ must be equivalent to the category of representations of a group H, which is later showed to be isomorphic to $G_k^\vee$. A thorough account of the proof can also be found in [6]. We remark that if k is a field of good characteristic, then parity sheaves on Gr_G are always perverse [7]. In this case, the category of parity sheaves corresponds via geometric Satake to tilting modules for $G_k^\vee$.

The Finkelberg–Mirković conjecture. Assume now that k is a field of characteristic p. There are several properties of the category $\mathrm{Rep}(G_k^\vee)$ which have no known counterpart on the geometric side. The Finkelberg–Mirković conjecture would explain how to construct the Frobenius twist functor geometrically.

Let W be the Weyl group of G and let $W^{\mathrm{ext}} := W \ltimes X$ be the extended affine Weyl group. Let $L(\lambda)$ denote the irreducible representation of $G_k^\vee$ with highest weight $\lambda \in X$. Let $\mathrm{Rep}_0^{\mathrm{ext}}(G_k^\vee)$ be the Serre subcategory generated by all $L(\lambda)$, for $\lambda \in W \cdot_p 0 \cap X_+$, where $\cdot_p$ denotes the p-dilated dot action of W^{ext}. For every $\lambda \in X$, the Frobenius twist $\mathrm{Fr}(L(\lambda))$ of $L(\lambda)$ is isomorphic to $L(p\lambda)$ and belongs to $\mathrm{Rep}_0^{\mathrm{ext}}(G_k^\vee)$.

Let $P_{(Iw)}(Gr_G, k)$ denote the category of perverse sheaves constructible with respect to the stratification of Iw-orbits

Conjecture (Finkelberg–Mirković). There exists an equivalence of categories $\mathcal{Q}$: $P_{(Iw)}(Gr_G, k) \to \mathrm{Rep}_0^{\mathrm{ext}}(G_k^\vee)$ making the following diagram commutative.

$$\begin{tikzcd} P_{G(\mathcal{O})}(Gr_G, k) \arrow[r, "For_{G(\mathcal{O})}"] \arrow[d, "\mathcal{S}"'] & P_{(Iw)}(Gr_G, k) \arrow[d, "\mathcal{Q}"'] \\ \mathrm{Rep}(G_k^\vee) \arrow[r, "\mathrm{Fr}"] & \mathrm{Rep}_0^{\mathrm{ext}}(G_k^\vee) \end{tikzcd}$$

Assuming the Finkelberg–Mirković conjecture, Achar and Riche have recently given a geometrical proof of Steinberg tensor product theorem [8].

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Torsion Explosion

Tom Gannon

Let k be some algebraically closed field, and let G denote some reductive group over k for which the characteristic of k is either zero or larger than the Coxeter number. If k has postive characteristic, a main insight in [1] is that some of the weights in Lusztig's conjecture can be reduced to studying a certain Serre quotient category of the category of representations of G known as *modular category* $\mathcal{O}_0$. Using this, in [3] and [4], Williamson showed that any bound for which Lusztig's conjecture holds for GL_n must be exponential in n. In this talk, we discuss these results and their broader context in this Arbeitsgemeinschaft.

1. Modular Category $\mathcal{O}_0$

The idea and first results of [1] state that the modular category $\mathcal{O}_0$ has many similar properties to the usual block $\mathcal{O}_0$ of the BGG category $\mathcal{O}$ defined when k is characteristic zero. The main result of this section is the theorem of Soergel, whose proof can be found in [1].

Theorem 1. *We have the following:*

(1) *The functor $\mathbb{V}$ defines an equivalence between the additive category of projective objects of the category $\mathcal{O}_0$ and the category of ungraded Soergel modules.*
(2) *The hypercohomology functor restricted to the full subcategory of parity sheaves on the complex flag manifold of $G^\vee$ with coefficients in k defines an equivalence between this category and (graded) Soergel modules.*
(3) *Under these equivalences, the character of an indecomposable projective object of $\mathcal{O}_0$ in the Grothendieck group of $\mathcal{O}_0$, which we identify with $\mathbb{Z}[W]$ via the basis of standard objects, is equivalently given by a 'graded stalk' of the associated indecomposable parity sheaf when the cohomological parameter is set to 1.*

The proof of theorem 1 is essentially uniform across all characteristics except those positive characteristics smaller than the Coxeter number. It also reduces the computation of the composition multiplicities of a given simple object $L_{x\cdot 0}$ in a Verma $\Delta_{y\cdot 0}$ as follows. By BGG reciprocity, we may equivalently compute the Verma flag multiplicity $(P_{x\cdot 0} : \Delta_{y\cdot 0})$. We may use the equivalence given by $\mathbb{V}$ to identify this indecomposable projective object with an indecomposable Soergel module labelled by x. This indecomposable Soergel module admits a graded lift, and therefore since hypercohomology $\mathbb{H}$ also gives an equivalence of categories, we may choose an object which maps via hypercohomology to this Soergel module– this will turn out to be an indecomposable parity sheaf. We then use the 'stalks of parity sheaves' map to compute the character of v in terms of the stalks of these parity sheaves. In this way, one can rephrase Theorem 1 as follows:

Theorem 2. *For k the characteristic of p larger than the Coxeter number, the k-parity sheaves agree with the k-intersection cohomology sheaves if and only if Lusztig's conjecture holds.*

2. Torsion Explosion

Using Theorem 2, Williamson proved the following theorem, which in particular shows that the expected bound on the characteristic of the field for which Lusztig's conjecture holds for some GL_n is substantially higher than expected:

Theorem 3. *('Torsion Explosion',* [3], [4]*) For all n, there exists some p_n for which the decomposition theorem fails for a field of characteristic p_n, so that in particular there exists some indecomposable parity sheaf P_w which is not equivalent to the associated intersection cohomology sheaf, and, moreover, the assignment $n \mapsto p_n$ is exponential in n.*

In the second part of this talk, we discuss the geometric proof of this result given in [4], and sketch the idea behind the proof as follows. The decomposition theorem holds in characteristic p if and only if all the ranks of certain intersection forms of a fixed Bott-Samuelson and some fiber have the same rank over $\mathbb{Q}$ as they do over a field of characteristic p. Therefore, one might hope for a situation by which the associated intersection form of some Bott-Samuelson resolution and some fiber is given by a 1×1 matrix with a nonzero coefficient. If one could find such a resolution, then any prime dividing an entry would be a prime for which Lusztig's conjecture fails. A geometric situation for which this arises is discussed in [4], and is known as the *miracle situation*. Williamson shows that there exists certain expressions for which the miracle situation holds and moreover gives the algorithm to compute the associated 1×1 matrix. In [3], he shows that a slightly more general class of these expressions can be used to show that, for $n \gg 0$, any prime dividing the n^{th} Fibonacci number must divide the intersection form, and thus is a prime for GL_n. Since the n^{th} Fibonacci number must have n distinct prime divisors [2], there is some prime larger than $n\log(n)$ for which Lusztig's conjecture fails for GL_n. Using more sophisticated expressions, one can show the exponential growth in the torsion explosion theorem.

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On Tilting Characters

Bregje Pauwels

Definitions and notation. Let G be a connected reductive algebraic group over an algebraically closed field of characteristic $p > 0$. For simplicity, we will assume that DG is simply connected. We write $\mathrm{Rep}(G)$ for the category of finite-dimensional (algebraic) G-modules. Choosing a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$, we denote by $\mathbb{X}$ the lattice of characters of T, and by $\mathcal{R} \subset \mathbb{X}$ the root system of (G, T). We write $\mathcal{R}^+ = \mathcal{R} \setminus \mathcal{R}(B)$ for the subset of positive roots, and the associated system of simple roots will be denoted $\mathcal{R}^s \subset \mathcal{R}^+$. Our choice of $\mathcal{R}^+$ determines a subset $\mathbb{X}^+$ of *dominant weights* in $\mathbb{X}$ and an order $\preceq$ on $\mathbb{X}$. We write

$$\mathbb{X}^+_{\mathrm{res}} = \{\lambda \in \mathbb{X} \mid \forall \alpha \in \mathcal{R}^s, 0 \leq \langle \lambda, \alpha^\vee \rangle < p\}$$

for the subset of *dominant restricted weights*. We fix a weight $\varsigma \in \mathbb{X}$ such that $\langle \varsigma, \alpha^\vee \rangle = 1$ for any $\alpha \in \mathcal{R}^s$.

For any dominant weight $\lambda \in \mathbb{X}^+$, we write ∇_λ for the *induced module* and Δ_λ for the *Weyl module* associated to λ. Thus Δ_λ admits a unique simple quotient L_λ, which is also the unique simple submodule of ∇_λ. The assignment $\lambda \mapsto L_\lambda$ induces a bijection between $\mathbb{X}^+$ and the set of all simple G-modules up to isomorphism.

Recall that, in characteristic p, we have the *Frobenius morphism* $\mathrm{Fr} : G \to G$. If $G = GL_n$ for example, Fr raises matrix entries to their p-th power. The *Frobenius twist* is then defined to be the functor $\mathrm{Fr}^* : \mathrm{Rep}(G) \to \mathrm{Rep}(G)$.

Motivation. The characters of induced modules and Weyl modules are well-understood by Weyl's character formula. Hence, to know the character of a simple module L, it suffices to express its class in the Grothendieck group in the basis $\{\Delta_\lambda,\ \lambda \in \mathbb{X}^+\}$ or the basis $\{\nabla_\lambda,\ \lambda \in \mathbb{X}^+\}$. This is a very hard problem; the dimensions and characters of the simple representations are unknown in general.

Changing the basis of the root system, or *tilting* the axes can be useful in this regard. This leads us to the notion of *tilting modules*. Below, we will classify all tilting modules, compute their characters in the case that $G = SL_2$, and explain why knowledge of tilting characters would imply knowledge of characters of simple modules.

Tilting objects in highest weight categories. There is a canonical structure of *highest weight category* on $\mathrm{Rep}(G)$, with weight poset $(\mathbb{X}^+, \preceq)$, standard objects $\{\Delta_\lambda,\ \lambda \in \mathbb{X}^+\}$ and costandard objects $\{\nabla_\lambda,\ \lambda \in \mathbb{X}^+\}$. In particular, we have that

$$\mathrm{Ext}^i(\Delta_\lambda, \nabla_\mu) = \begin{cases} k \text{ if } i = 0 \text{ and } \lambda = \mu \\ 0 \text{ otherwise.} \end{cases}$$

In that sense, the induced and Weyl modules form a dual basis for the derived category $D^b(\mathrm{Rep}(G))$.

Recall that a *tilting object* in a highest weight category is an object which admits both a filtration by standard objects and a filtration by costandard objects. A tilting object in $\mathrm{Rep}(G)$ is also called a *tilting G-module*. For $\lambda \in \mathbb{X}^+$, we write $(M : \nabla_\lambda)$ for the number of times that ∇_λ occurs in a(ny) filtration by costandard objects of the tilting module M . The multiplicities $(M : \Delta_\lambda)$ are defined similarly by considering standard filtrations. In the Grothendieck group, we then have

$$[M] = \sum_{\lambda \in \mathbb{X}^+} (M : \nabla_\lambda) \cdot [\nabla_\lambda] \quad \text{and} \quad [M] = \sum_{\lambda \in \mathbb{X}^+} (M : \Delta_\lambda) \cdot [\Delta_\lambda].$$

Note that the coefficients in the expansion of tilting modules in the bases of induced modules and Weyl modules are nonnegative. This allows us to use combinatorial tools to study the multiplicities (and hence the characters) of tilting modules.

Classification of tilting modules. Every tilting module can be written as a direct sum of indecomposable tilting modules, so it suffices to describe the indecomposable ones. The classification of tilting objects in a highest weight category (see [5, App. A]), applied to the category $\mathrm{Rep}(G)$, leads to the following

Theorem 1 (Ringel, Donkin). *For any $\lambda \in \mathbb{X}^+$, there exists an indecomposable tilting G-module T_λ such that $(T_\lambda : \nabla_\lambda) = 1$ and $(T_\lambda : \nabla_\mu) = 0$ unless $\mu \preceq \lambda$. The assignment $\lambda \mapsto T_\lambda$ induces a bijection between $\mathbb{X}^+$ and the set of all indecomposable tilting G-modules up to isomorphism.*

Tensor products of tilting modules. The following theorem is due to Mathieu.

Theorem 2. *For any tilting G-modules M, N, the tensor product $M \otimes N$ is tilting.*

The *tilting tensor formula* ([2, §II.E.9]), due to Donkin, is the tilting analogue of Steinberg's tensor formula:

Theorem 3. *Assume $p \geq 2h - 2$. For any $\lambda \in (p-1)\varsigma + \mathbb{X}^+_{res}$ and any $\mu \in \mathbb{X}^+$,*

$$T_{\lambda + p\mu} \simeq T_\lambda \otimes \mathrm{Fr}^*(T_\mu).$$

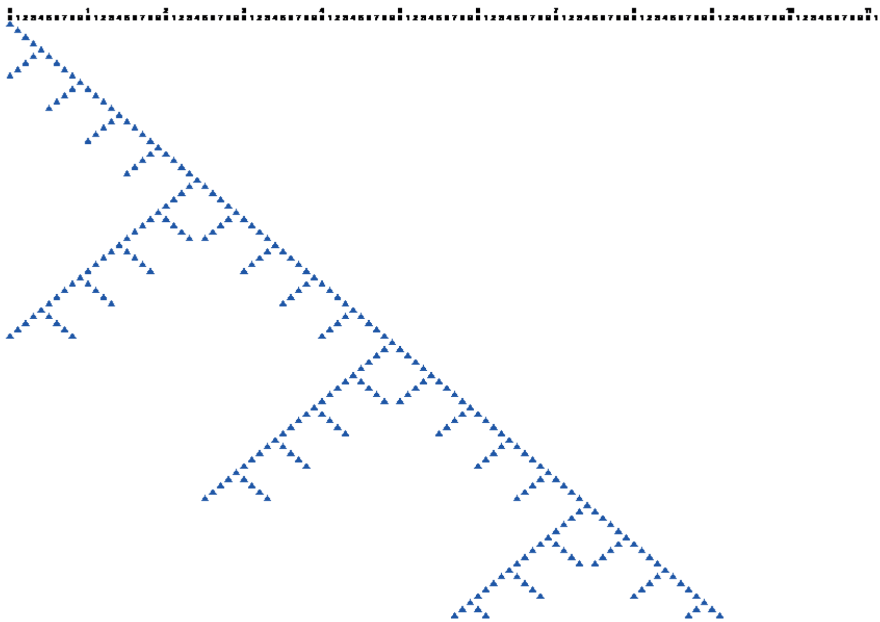
Recall that Steinberg's formula's magic allows us to reduce the determination of characters of simple modules to weights in a finite number of closures of alcoves. Unfortunately, the analogue does not work for tilting modules: the tilting tensor formula reduces the determination of characters of indecomposable tilting modules to weights in infinitely many closures of alcoves, unless G is of type A_1.

Example. For $G = SL_2$ we can describe all indecomposable tilting modules. We let $\mathbb{X} = \mathbb{Z}$, $\mathbb{X}^+ = \mathbb{Z}_{\geq 0}$, $\mathbb{X}^+_{\mathrm{res}} = \{0, 1, \ldots, p-1\}$ and $\varsigma = 1$. By the tilting tensor formula, the problem reduces to finding T_i for $0 \leq i \leq 2p-2$. If $0 \leq i \leq p-1$, we have $\nabla_i = \Delta_i = L_i = k[X,Y]_i$, the space of homogeneous polynomials of degree i. It follows that $T_i = k[X,Y]_i$. For $p \leq i \leq 2p-2$, using translation functors we find a short exact sequence

$$\nabla_{2p-2-i} \hookrightarrow T_i \twoheadrightarrow \nabla_i.$$

Hence we can explicitly compute all multiplicities $(T_i : \nabla_j)$, or equivalently all characters of tilting modules T_i, for $i \in \mathbb{Z}_{\geq 0}$. This is the only semi-simple group for which all tilting characters are known. We refer to [1] and [3] for more details.

After the talk, Joel Gibson added a tool to his wonderful website [4, LieVis] to visualize the multiplicities $(T_i : \nabla_j)$. The following picture, made in LieVis, shows the multiplicities of ∇_j in T_i for $p = 5$ (a triangle indicates the multiplicity is 1).



From tilting characters to simple characters. Let $p \geq 2h-2$. The following theorem shows that, if we knew the characters of the indecomposable tilting G-modules, or equivalently all multiplicities $(T_\mu : \nabla_\lambda)$, then we could in theory compute the characters of simple G-modules. For more details, we refer to [6] and the references therein. Consider the subset of $\mathbb{X}^+$ defined by

$$\mathbb{X}^+_{\mathrm{bb}} = \{\lambda \in \mathbb{X}^+ \mid \langle \lambda, \alpha^\vee \rangle \leq (p-1)\langle \varsigma, \alpha^\vee \rangle \text{ for every dominant short root } \alpha\}.$$

Note that $\mathbb{X}^+_{\mathrm{res}} \subset \mathbb{X}^+_{\mathrm{bb}}$ and that $\mathbb{X}^+_{\mathrm{bb}}$ is an ideal for $\preceq$.

Theorem 4. *There is an (explicit) bijection* $(-)^\blacktriangle : \mathbb{X}^+ \xrightarrow{\sim} (p-1)\varsigma + \mathbb{X}^+$ *such that*

$$[\Delta_\lambda : L_\mu] = (T_{\mu^\blacktriangle} : \nabla_\lambda)$$

for any $\lambda, \mu \in \mathbb{X}^+_{bb}$.

Tilting modules for quantum groups. The following theorem, due to Soergel, expresses the characters of tilting modules for quantum groups at roots of one in terms of the anti-spherical Kazhdan-Lusztig polynomials $n_{A,B}$.

Theorem 5. $(T_A : \nabla_B) = n_{B,A}(1)$

We refer to [7] for details and motivation, and to [8] for a proof.

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The Categorical Conjecture

Jonathan Gruber

In their landmark monograph [RW18], Simon Riche and Geordie Williamson proposed two conjectures relating the category of tilting modules in the principal block $\mathrm{Rep}_0(G)$ of a reductive algebraic group G to the diagrammatic Hecke category $\mathcal{D}$ corresponding to the affine Weyl group W of G.

In the first one, here called the *categorical conjecture*, they propose that the principal block of G should be a module category over $\mathcal{D}$, in such a way that the monoidal generators of $\mathcal{D}$ act via so-called *wall-crossing functors*.

In the second one, called the *numerical conjecture*, they conjecture that the multiplicities in Weyl filtrations of indecomposable tilting modules should be given by the values at one of certain *anti-spherical p-Kazhdan-Lusztig polynomials*.

(Both conjectures are now known to be true.)

In this talk, we first introduce some background material in order to then state the precise versions of the two conjectures.

We also explain a natural $\mathbb{Z}[W]$-module structure on the Grothendieck group of $\mathrm{Rep}_0(G)$, which serves as a motivation for the categorical conjecture.

Finally, we discuss some consequences of the categorical conjecture and explain why the latter implies the numerical conjecture.

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An Iwahori–Whittaker presentation of the Satake Category

Valentin Gouttard

Groups. Let $\mathbb{F}$ be an algebraically closed field of positive characteristic $p > 0$. We consider a connected reductive algebraic group G over $\mathbb{F}$; fix a maximal torus T and a Borel subgroup $T \subseteq B^+$. Let U^+ denote the unipotent radical of B^+. The character lattice $\mathrm{X}^*(T)$ will be denoted $\mathbf{X}$, and the cocharacter lattice $\mathrm{X}_*(T)$ will be denoted $\mathbf{X}^\vee$. Let Δ be the set of roots, $\Delta^\vee$ the set of coroots. Our choice of Borel subgroup B^+ induces a subset of positive roots: let Δ^+ be the roots that appear in $\mathrm{Lie}(U^+)$. We then get a subset Δ^s of simple roots. For any root α, we choose and fix an isomorphism $u_\alpha : \mathbb{G}_{a,\mathbb{F}} \xrightarrow{\sim} U_\alpha$.

Recall that we have a pairing $\langle -,- \rangle : \mathbf{X}^\vee \times \mathbf{X} \to \mathbb{Z}$. We define dominant and strictly dominant cocharacters as follows:

$$\mathbf{X}^\vee_+ := \{\lambda \in \mathbf{X}^\vee \mid \langle \lambda, \alpha \rangle \geq 0 \;\forall \alpha \in \Delta^s\}$$

$$\mathbf{X}^\vee_{++} := \{\lambda \in \mathbf{X}^\vee \mid \langle \lambda, \alpha \rangle > 0, \;\forall \alpha \in \Delta^s\}.$$

We assume that there exists a cocharacter ς such that $\langle \varsigma, \alpha \rangle = 1$ for any simple root α (this is true for example if G is a semisimple group of adjoint type; in which case we can take $\varsigma = \rho := \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$). Note that ς may not be unique. Under this assumption, we have $\mathbf{X}^\vee_{++} = \varsigma + \mathbf{X}^\vee_+$.

Affine Grassmaniann. We set $\mathscr{K} = \mathbb{F}((z))$ and $\mathscr{O} = \mathbb{F}[[z]]$ and consider the ind-group scheme $G_{\mathscr{K}}$ and its group subscheme $G_{\mathscr{O}}$. We have a natural morphism $ev_0 : G_{\mathscr{O}} \to G$ induced by $\mathbb{F}[[z]] \to \mathbb{F}$, $z \mapsto 0$; let I^+ be the inverse image of B^+ under ev_0, and I^+_u the inverse image of U^+.

Define the affine Grassmannian $Gr := G_{\mathscr{K}}/G_{\mathscr{O}}$; one can show that Gr is represented by an ind-projective ind-scheme of ind-finite type (see for example [Z, Theorem 1.2.2]). To any cocharacter $\lambda \in \mathbf{X}^\vee$, we can associate a point in Gr denoted L_λ. The affine Grassmaniann is the union of $G_{\mathscr{O}}$-orbits: set $Gr^\lambda := G_{\mathscr{O}} \cdot L_\lambda$, we then have $Gr = \bigsqcup_{\lambda \in \mathbf{X}^\vee_+} Gr^\lambda$. One can show that the orbits Gr^λ are irreducible quasi-projective schemes of finite type, of dimension $\langle \lambda, 2\rho \rangle$. We let $j_\lambda : Gr^\lambda \hookrightarrow Gr$ be the embedding.

$G_{\mathscr{O}}$-equivariant sheaves and the Geometric Satake Equivalence. Let ℓ be a prime number $\neq p$. In these notes, $\Bbbk$ will denote one of the following ring: a finite extension of $\mathbb{Q}_\ell$, the ring of integers in such an extension, or a finite field of characteristic ℓ. We consider the $G_{\mathscr{O}}$-equivariant derived category of (étale) $\Bbbk$-sheaves on Gr and the perverse subcategory $P_{G_{\mathscr{O}}}(Gr, \Bbbk)$; this identifies with the heart of the t-structure on $D^b_{G_{\mathscr{O}}}(Gr, \Bbbk)$. Set

$$\mathcal{I}_!(\lambda, \Bbbk) := {}^p\mathscr{H}^0((j_\lambda)_! \underline{\Bbbk}_{Gr^\lambda}[\langle \lambda, 2\rho \rangle]), \quad \mathcal{I}_*(\lambda, \Bbbk) := {}^p\mathscr{H}^0((j_\lambda)_* \underline{\Bbbk}_{Gr^\lambda}[\langle \lambda, 2\rho \rangle]).$$

The objects $\mathcal{I}_!(\lambda, \Bbbk)$ and $\mathcal{I}_*(\lambda, \Bbbk)$ are then perverse. For simplicity here, we assume that $\Bbbk$ is a field. We then have the following results.

Proposition 1. *The category $P_{G_{\mathscr{O}}}(Gr,\Bbbk)$ is a highest weight category, with associated weight poset $(\mathbf{X}^\vee_+,\preceq)$, standard objects $\{\mathcal{I}_!(\lambda,\Bbbk) \mid \lambda \in \mathbf{X}^\vee_+\}$ and costandard objects $\{\mathcal{I}_*(\lambda,\Bbbk) \mid \lambda \in \mathbf{X}^\vee_+\}$. Moreover, endowed with the convolution product $(-) \star^{G_{\mathscr{O}}} (-)$ this is a monoidal category.*

Theorem 2 (Geometric Satake Equivalence, [MV])**.** *We have an equivalence of monoidal categories*

$$(P_{G_{\mathscr{O}}}(Gr,\Bbbk),(-)\star^{G_{\mathscr{O}}}(-)) \xrightarrow{\sim} (\mathrm{Rep}(G^\vee_\Bbbk),(-)\otimes_\Bbbk(-)).$$

Here, the right hand side denotes the category of rational representations of the $\Bbbk$-group scheme which is Langlands dual of G, endowed with the monoidal product given by the usual tensor product.

Iwahori–Whittaker category. We assume that there exists a primitive p-th root of unity in $\Bbbk$, and we fix one. This choice allows us to define a character $\psi : \mathbb{F}_p \to \Bbbk^*$ which in turn defines an étale $\Bbbk$-local system of rank 1 on $\mathbb{G}_{a,\mathbb{F}}$, which we denote $\mathscr{L}^\Bbbk_\psi$.

Consider the following composition

$$\chi_{I_u^+} : I_u^+ \xrightarrow{ev_0} U^+ \twoheadrightarrow U^+/[U^+,U^+] \xleftarrow[\prod u_\alpha]{\sim} \prod_{\alpha\in\Delta^s} \mathbf{G}_{a,\mathbb{F}} \xrightarrow{+} \mathbf{G}_{a,\mathbb{F}}.$$

We define the derived category $D^b_{\mathcal{IW}}(Gr,\Bbbk)$ of $(I_u^+,\chi^*_{I_u^+}(\mathscr{L}^\Bbbk_\psi))$-equivariant objects as in [AR]: this is the full subcategory of $D^b(Gr,\Bbbk)$ whose objects are those complexes $\mathscr{F}$ such that $a^*_{I_u^+}(\mathscr{F}) = \chi^*_{I_u^+}(\mathscr{L}^\Bbbk_\psi) \boxtimes^L_\Bbbk \mathscr{F}$ (here, $a_{I_u^+} : I_u^+ \times Gr \to Gr$ denotes the induced action morphism). This is a triangulated category, we can consider a natural perverse t-structure whose heart will be denoted $P_{\mathcal{IW}}(Gr,\Bbbk)$. (A little care should be taken here considering that we are working on an ind-scheme acted on by a pro-group, but we ignore such technicalities here.)

We set $X_\lambda := I^+\cdot L_\lambda = I_u^+\cdot L_\lambda$; we have $Gr = \bigsqcup_{\lambda\in\mathbf{X}^\vee} X_\lambda$. It turns out that not all I^+-orbits can support an $(I_u^+,\chi^*_{I_u^+}(\mathscr{L}^\Bbbk_\psi))$-equivariant local system (and thus an $(I_u^+,\chi^*_{I_u^+}(\mathscr{L}^\Bbbk_\psi))$-equivariant object):

Lemma 3. [BGMRR, Lemma 3.3] *There exists an $(I_u^+,\chi^*_{I_u^+}(\mathscr{L}^\Bbbk_\psi))$-equivariant local system on the orbit X_λ if and only if $\lambda \in \mathbf{X}^\vee_{++}$.*

For any $\lambda \in \mathbf{X}^\vee_{++}$, we can then define two perverse objects by letting $\Delta^{\mathcal{IW}}_\lambda(\Bbbk) := (j_{X_\lambda})_!(\mathscr{L}^\Bbbk_\psi(\lambda))[\dim(X_\lambda)]$ and $\nabla^{\mathcal{IW}}_\lambda(\Bbbk) := (j_{X_\lambda})_*(\mathscr{L}^\Bbbk_\psi(\lambda))[\dim(X_\lambda)]$. Here, j_{X_λ} denotes the embedding $X_\lambda \hookrightarrow Gr$ and $\mathscr{L}^\Bbbk_\psi(\lambda)$ denotes the (unique) $(I_u^+,\chi^*_{I_u^+}(\mathscr{L}^\Bbbk_\psi))$-equivariant local system on X_λ. An important feature of this new category is the following: the natural realization functor $D^bP_{\mathcal{IW}}(Gr,\Bbbk) \to D^b_{\mathcal{IW}}(Gr,\Bbbk)$ is an equivalence of categories (the proof goes as in [BGS, §3.2 and §3.3]). Moreover, if $\Bbbk$ is a field, we have nice structural properties:

Lemma 4. [BGMRR, Corollary 3.6] *If $\Bbbk$ is a field, the category $P_{\mathcal{IW}}(Gr,\Bbbk)$ admits a highest weight structure with weight poset $(\mathbf{X}^\vee_{++},\preceq)$, standard objects $\{\Delta^{\mathcal{IW}}_\lambda(\Bbbk) \mid \lambda \in \mathbf{X}^\vee_{++}\}$ and costandard objects $\{\nabla^{\mathcal{IW}}_\lambda(\Bbbk) \mid \lambda \in \mathbf{X}^\vee_{++}\}$.*

Main Result. An object $\mathscr{F}$ in $D^b_{\mathcal{IW}}(Gr, \Bbbk)$ is $G_{\mathscr{O}}$-equivariant on the right, so we can convolve $\mathscr{F}$ on the right with an object $\mathscr{G}$ in $D^b_{G_{\mathscr{O}}}(Gr, \Bbbk)$. As the Iwahori–Whittaker condition is an "equivariance on the left" condition, one can check that the convolution product $\mathscr{F} \star^{G_{\mathscr{O}}} \mathscr{G}$ is in $D^b_{\mathcal{IW}}(Gr, \Bbbk)$. Thus the Iwahori–Whittaker category is a right module over the category $D^b_{G_{\mathscr{O}}}(Gr, \Bbbk)$. Consider the object $\Delta^{\mathcal{IW}}_{\varsigma}(\Bbbk)$.

Let us define a functor $\Phi : D^b_{G_{\mathscr{O}}}(Gr, \Bbbk) \to D^b_{\mathcal{IW}}(Gr, \Bbbk)$ by setting

$$\Phi(\mathscr{F}) := \Delta^{\mathcal{IW}}_{\varsigma}(\Bbbk) \star^{G_{\mathscr{O}}} \mathscr{F}.$$

It is proved in [BGMRR, Lemma 3.8] that this functor is t-exact for the perverse t-structure; we denote $\Phi^0 : P_{G_{\mathscr{O}}}(Gr, \Bbbk) \to P_{\mathcal{IW}}(Gr, \Bbbk)$ its restriction (note that here, the claim is true without the assumption that $\Bbbk$ is a field). The main result of [BGMRR] is then:

Theorem 5. [BGMRR, Theorem 3.9] *The functor Φ^0 is an equivalence categories, mapping $\mathcal{I}_!(\lambda, \Bbbk)$ to $\Delta^{\mathcal{IW}}_{\varsigma+\lambda}(\Bbbk)$ and $\mathcal{I}_*(\lambda, \Bbbk)$ to $\nabla^{\mathcal{IW}}_{\varsigma+\lambda}(\Bbbk)$.*

In particular, if $\Bbbk$ is a field, Φ^0 is an equivalence of highest weight categories. We can get back to our previous somehow-artificial-remark on the weight posets: going from the Satake to the Iwahori–Whittaker perverse category (via Φ^0), we shift the weight poset by ς. Composing the previous equivalence with the geometric Satake equivalence and passing to derived categories, we get a "derived version" of the latter equivalence

$$D^b\mathrm{Rep}(G^\vee_{\Bbbk}) \xrightarrow{\sim} D^b P_{\mathcal{IW}}(Gr, \Bbbk) \cong D^b_{\mathcal{IW}}(Gr, \Bbbk).$$

Parity and tilting objects. If $\Bbbk$ is a field, we can define even, odd and parity objects in the Iwahori–Whittaker category. We say that an object $\mathscr{F} \in D^b_{\mathcal{IW}}(Gr, \Bbbk)$ is even (resp. odd) if its restriction and corestriction to any strata X_λ with $\lambda \in \mathbf{X}^\vee_{++}$ is concentrated in even (resp. odd) degrees. We say that $\mathscr{F}$ is parity if it is isomorphic to a direct sum of even and odd objects.

We can also define even, odd and parity objects in $D^b_{(G_{\mathscr{O}})}(Gr, \Bbbk)$. It is known ([JMW, Theorem 4.6]) that we have parity sheaves in this category: for any $\lambda \in \mathbf{X}^\vee_+$ there exists an indecomposable parity object $\mathcal{E}_\lambda$ supported on $\overline{Gr^\lambda}$ and whose restriction to Gr^λ is $\underline{\Bbbk}_{Gr^\lambda}[\langle \lambda, 2\rho \rangle]$.

Theorem 6. [BGMRR, Theorem 4.10] *For any $\lambda \in \mathbf{X}^\vee_+$ and any $n \in \mathbb{Z}$, the perverse sheaf ${}^p\mathscr{H}^n(\mathcal{E}_\lambda)$ is tilting in the highest weight category $P_{G_{\mathscr{O}}}(Gr, \Bbbk)$.*

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Smith-Treumann Theory

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The goal of my talk at the Arbeitsgemeinschaft was to give an account of Smith theory for sheaves, as initiated by D. Treumann, with literature [Tre19], [RW19] and [Wil19]. In the following couple of pages, I'll summarize some highlights from this theory. I've tried to kept it as "generic" and non-technical as possible, to highlight structural features, rather than particular technical issues of a given model, perhaps at the slight expense of precision. The version I present here also differ somewhat from the sources. I'll remark on this as we go along, and at the end.

Classical Smith theory, named after P. A. Smith, whose works date from the 1930's, is a collection of results stating relations between mod ℓ invariants of a space X and those of the fixed-points X^{μ_ℓ}, where μ_ℓ is a finite group of prime order ℓ acting on X. (There are related characteristic zero results where μ_ℓ is replaced by the circle $T = S^1$, or, for the daring, its ℓ-torsion points μ_{ℓ^∞}.)

Smith's ideas have been very influential in the intervening 80+ years. It was recast in the 1960s and 1970s in the work of A. Borel, D. Quillen, and others, e.g., through the localization theorem. This again fed into fundamental conjectures in homotopy theory by G. Segal and D. Sullivan relating μ_ℓ–fixed-points and homotopy μ_ℓ–fixed-points, proved in the mid 80s by G. Carlsson and H. Miller, after 15 years of intense interest.

Let X be, say, a finite T-CW complex, for T the circle S^1, and k a field of characteristic ℓ. Write $H^*_T(-;k)$ for Borel equivariant cohomology with coefficients in k. Recall that a version of the classical localization theorem says that the restriction map

$$H^*_T(X;k) \to H^*_T(X^{\mu_\ell};k),$$

which is a map of $H^*_T(pt;k)$–algebras via the map p to a point, becomes an isomorphism after inverting the degree 2 class $u \in k[u] \cong H^*_T(pt;k)$, i.e.,

$$H^*_T(X;k)[u^{-1}] \xrightarrow{\cong} H^*_T(X^{\mu_\ell};k)[u^{-1}] \cong H^*(X^{\mu_\ell};k) \otimes_k k[u,u^{-1}].$$

Many statements of Smith theory follow from this formula. For example one sees that if X is mod ℓ acyclic, i.e., $H^*(X;k)$ one-dimensional over k, then the same holds for X^{μ_ℓ}, as $H^*_T(X;k) \xleftarrow{\cong} H^*_T(pt;k) \cong k[u]$. Similarly one sees that the μ_ℓ–fixed-points of a mod ℓ homology sphere is again a mod ℓ homology sphere or the empty set ("the (-1)-sphere"), by considering the pair (CX, X), for CX the cone on X.

The Treumann version of the localization theorem, as further refined by Riche-Williamson, is a similar result for the whole T–equivariant bounded derived category of sheaves $D^b_T(X;k)$, or alternatively T–equivariant constructible sheaves. Here X can either be as before, or a real subanalytic variety (the setup of Treumann [Tre19]) or a finite type T-scheme over a field $\mathbb{F}$ of characteristic $p \neq \ell$, with $T = \mathbb{G}_m$, and equipped with the étale topology (the setup of [RW19]).

For this, consider the restriction map

$$D^b_T(X;k) \xrightarrow{i^*} D^b_T(X^{\mu_\ell};k)$$

between triangulated (or ∞–) categories. Note that this is a morphism under $D^b_T(pt;k)$ via p^*, where p is the map to a point. We can similarly consider a morphism $(u : k \to k[2]) \in D^b_T(pt;k)$ representing the class u from before. Inverting u in this setting translates into "killing the cone $\operatorname{cofib}(u)$", i.e., forming the Verdier quotient with respect to the thick tensor ideal generated by $\operatorname{cofib}(u)$ in $D^b_T(X;k)$ and $D^b_T(X^{\mu_\ell};k)$ respectively, via p^*. Let us denote forming this quotient by $(-)[u^{-1}]$, and abbreviate "thick tensor ideal" to just ideal. In this formulation the theorem becomes:

Theorem 1 ("The localization theorem for sheaves")**.** *Restriction induces an equivalence of triangulated (or ∞–) categories*

$$i^* : D^b_T(X;k)[u^{-1}] \xrightarrow{\simeq} D^b_T(X^{\mu_\ell};k)[u^{-1}].$$

Likewise $i^!$ also descends to the quotient, where it agrees with i^*.

Note that the quotient category is 2-periodic, with periodicity induced by multiplication by u (an element in homological grading -2). In particular the quotient is not the bounded derived category of any ordinary ring. (It is one over the Tate fixed-point spectrum k^{tT}, though.) The quotient categories, should be thought of as a "Tate construction" applied to the original categories (and this can indeed be made precise!). E.g.,

$$\operatorname{Ext}^*_{D^b_T(pt;k)[u^{-1}]}(k,k) = k[u, u^{-1}]$$

It can also be described as a "singularity category" or "stable module category", of dualizable objects modulo compact objects, a sort of "category at infinity".

One of the wonderful things about the isomorphism in Theorem 1 is that it is as natural in X as can be, in the sense that it commutes with all the usual functors we consider:

Theorem 2 ("Localization commutes with all operations")**.** *Let $f : X \to Y$ be a T–equivariant morphism in one of the categories from earlier. Then f_*, f^*, $f_!$, $f^!$ and Verdier duality $\mathbb{D}$ preserve the property of being in the ideal generated by* $\operatorname{cofib}(u)$*, so also induce functors after inverting u, and the usual adjunctions continue to hold in the quotient.*

Furthermore for F either f_*, f^*, $f_!$ or $f^!$, the following diagram commute

$$\begin{tikzcd} D^b_T(X;k)[u^{-1}] \arrow[r, dash, "F"] \arrow[d, "i^*"] & D^b_T(Y;k)[u^{-1}] \arrow[d, "i^*"] \\ D^b_T(X^{\mu_\ell};k)[u^{-1}] \arrow[r, dash, "F"] & D^b_T(Y^{\mu_\ell};k)[u^{-1}] \end{tikzcd}$$

(with F going in the appropriate direction), and likewise for $F = \mathbb{D}$ with $Y = X$.

Before sketching the proof of these two theorems, let us make a note about the ideal generated by $\mathrm{cofib}(u)$. Upon restriction to $k\mu_\ell$, the chain complex $\mathrm{cofib}(u)$ can be modelled by

$$\cdots \to 0 \to k\mu_\ell \xrightarrow{1-g} k\mu_\ell \to 0 \to \cdots ,$$

non-zero in degree 1 and 0, and g a generator of μ_ℓ. In particular, the stalk of $p^*(\mathrm{cofib}(u))$ at $x \in X^{\mu_\ell}$ will be a perfect complex of $k\mu_\ell$–modules. As this is perserved under tensor products and summands, the same is true for any sheaf in the ideal generated $\mathrm{cofib}(u)$. Theorems 1 and 2 thus imply similar statements where we quotient out by the, a priori larger, subcategory given by all sheaves $\mathcal{F}$ where the stalk at $x \in X^{\mu_\ell}$ is a perfect complex of $k\mu_\ell$–modules. This was what was considered in [Tre19] and [RW19]. $D^b_T(X^{\mu_\ell};k)$ modulo this, a priori larger, subcategory is called $\mathrm{Perf}(X^{\mu_\ell};\mathcal{T})$ in [Tre19] and $\mathrm{Sm}(X^{\mu_\ell},k)$ in [RW19]. Under suitable niceness assumptions on X the two ideals coincide, but the best-possible result in this direction is not entirely clear to us.

We will now sketch the proofs of the two theorems.

Sketch of proof of Theorem 1. We first prove the claim when $X^{\mu_\ell} = \emptyset$, i.e., if the T–action is "free at ℓ". Here the statement becomes that any sheaf $\mathcal{F} \in D^b_T(X;k)$ lies in the ideal generated by $\mathrm{cofib}(u)$. For this, one first observes that in this case $D^b_T(X;k) \cong D^b(X/T;k)$, as the action is "free at ℓ" (here one appeals to a property equivariant derived category in the relevant setting). In particular $\mathrm{Ext}^*_{D^b_T(X;k)}(\mathcal{F},\mathcal{F})$ is concentrated in only finitely many dimensions (a property of X/T). Hence, for some n, u^n maps to zero under the natural map

$$\mathrm{Ext}^*_{D^b_T(pt;k)}(k,k) \xrightarrow{p^*} \mathrm{Ext}^*_{D^b_T(X;k)}(\mathcal{F},\mathcal{F})$$

Said in other words, the map $\mathcal{F} \xrightarrow{u^n} \mathcal{F}[2n]$ is zero, and hence $\mathrm{cofib}(u^n) \otimes \mathcal{F} \cong \mathcal{F}[1] \oplus \mathcal{F}[2n]$. But $\mathrm{cofib}(u^n)$ lies in the ideal (as it can be constructed by iterated cofibers starting with $\mathrm{cofib}(u)$), and hence so does $\mathcal{F}$, as wanted.

For the general case, we consider the recollement

$$\begin{tikzcd} D^b_T(X^{\mu_\ell};k) \arrow[r, "i_*"] & D^b_T(X;k) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D^b_T(X \setminus X^{\mu_\ell};k) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

Here we need to see that the image under j_* of the right-hand term lies in the ideal generated by $\mathrm{cofib}(u)$. We have already seen that $D^b_T(X \setminus X^{\mu_\ell};k)$ equals the ideal, so this statement is covered by the first part of Theorem 2, which says that all the functors preserve the property of being in this ideal—we will sketch the proof of Theorem 2 below.

The last statement is similarly a consequence of the first part of Theorem 2. We need to see that the cofiber of $i^!\mathcal{F} \to i^*\mathcal{F}$ lies in the ideal. Recall that by six functor formalism 'localization triangle' we have the cofibration sequence $i_! i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F}$, which upon applying i^* gives a cofibration sequence $i^! \mathcal{F} \to i^* \mathcal{F} \to i^* j_* j^* \mathcal{F}$, as $i^* i_! = i^* i_* = 1$. But $i^* j_* j^* \mathcal{F}$ lies in the ideal generated generated by cofib(u), for the same reason as before: $j^*\mathcal{F}$ lies in $D^b_T(X \setminus X^{\mu_\ell}; k)$, and is particular in the ideal, and being in the ideal is preserved by $i^* j_*$ by the first half of Theorem 2. Hence $i^!$ and i^* agree on the quotient. (Compare also [Tre19, Sec. 4.2] and [RW19, Prop. 2.6].) □

Sketch of proof of Theorem 2. First note that $\mathbb{D}$ preserves the ideal generated by cofib(u). Namely, for $p : X \to pt$, we have that

$$\mathbb{D}(\mathrm{cofib}(u) \otimes \mathcal{F}) = \mathrm{map}(\mathrm{cofib}(u) \otimes \mathcal{F}, p^!(k)) \cong \mathrm{map}(\mathrm{cofib}(u), \mathbb{D}\mathcal{F}) \cong \mathrm{cofib}(u) \otimes \mathbb{D}\mathcal{F}[-1]$$

It is obvious that f^* preserves the ideal generated by cofib(u), as pullback is functorial, and hence the same is true for $f^!$ using that $= \mathbb{D} f^! = f^* \mathbb{D}$. That $f_!$ preserves the ideal follows from the projection formula $f_!(\mathcal{F} \otimes f^*(\mathcal{G})) \cong f_!(\mathcal{F}) \otimes \mathcal{G}$, with $\mathcal{G} = \mathrm{cofib}(u)$, under the assumptions when this formula holds, e.g., finite covering dimension of Y, and again this implies the same for f_*, by Verdier duality. This concludes the proof of the first part of Theorem 2 (used in the proof of Theorem 1).

Let us now check that the diagram commutes in all cases. For $F = \mathbb{D}$, we need to see that the cofiber of $\mathbb{D} i^* \mathcal{F} \cong i^! \mathbb{D} \mathcal{F} \to i^* \mathbb{D} \mathcal{F}$ is in the ideal generated by cofib(u), which follows by the last part of Theorem 1.

That the diagram commutes for $F = f^*$ is obvious as $(-)^*$ distributes over composition. The diagram also commutes for $F = f_!$ as $i^* f_! \xrightarrow{\cong} f_! i^*$ by base change. The statements for $F = f^!$ and $F = f_*$ now follows by Verdier duality, as Verdier duality transforms f^* into $f^!$ and $f_!$ into f_*. □

We end with a few remarks. As noted I've stated things a bit different from the original papers in this note, to get formulations closer to the original localization theorem in equivariant cohomology. In particular I've stated the main result as an equivalence of categories. Furthermore, as explained above, I'm also quotienting by a different (potentially smaller) subcategory than the one used in [Tre19] and [RW19] (though in practice probably often equivalent)—in my talk at the Arbeitsgemeinschaft I only gave a vague comment that something like that should be true, mumbling something about finiteness. This led to some confusion and follow-up conversations with Gurbir Dhillon and Geordie Williamson. A version indeed turns out to the true, and the above proof sketch follows that rute, thanks to those conversations.

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Smith–Treumann Theory and the Linkage Principle

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In this talk one of the main results of [2] was explained: the geometric realisation of the linkage principle through the Smith–Treumann localisation to fixed points.

Let $\check{G}$ be a reductive group over an algebraically closed field k of characteristic ℓ and fix a maximal torus and a Borel subgroup $\check{T} \leq \check{B} \leq \check{G}$. Let $\check{X}$ be the characters of $\check{T}$ and $\check{X}^+$ the dominant characters corresponding to the opposite Borel $\check{B}^+$. The extended affine Weyl group is denoted by $W_{\text{aff}} := W \ltimes \check{X}$. The linkage principle (due to Andersen) states that we have a decomposition of categories

$$\mathrm{Rep}_k(\check{G}) = \bigoplus_{\gamma \in \check{X}/W_{\text{aff}}, \bullet_\ell} \mathrm{Rep}_k^\gamma(\check{G}) \tag{1}$$

where $\mathrm{Rep}_k^\gamma(\check{G}) = \langle L(\lambda) \mid \lambda \in \gamma \rangle$ and $L(\lambda)$ is the irreducible representation of highest weight λ (or zero if λ is not dominant). The $\bullet_\ell$-action is the ℓ-dilated dot action: $wt_\lambda \bullet_\ell \mu := w(\mu + \ell\lambda + \rho) - \rho$.

Let G be the Langlands dual group of $\check{G}$ defined over an algebraically closed field $\mathbb{F}$ of characteristic $p \neq \ell$. We put $\mathcal{K} := \mathbb{F}((z))$ and $\mathcal{O} := \mathbb{F}[[z]]$ and we define the affine Grassmannian $\mathcal{G}r_G$ as the quotient $G_\mathcal{K}/G_\mathcal{O}$. It is an ind-scheme which is ind-projective and of ind-finite type. Let $I^+ \leq G_\mathcal{O}$ be the Iwahori subgroup corresponding to the unipotent radical of B^+ under the evaluation map $\mathrm{ev}_{z=0}\colon G_\mathcal{O} \to G$. The I^+ orbits on $\mathcal{G}r_G$ are parametrised by the characters $\check{X}$ and for $\lambda \in \check{X}$ we denote such an orbit by $\mathcal{G}r_{G,\lambda}$.

In previous talks, the following equivalences of categories were explained:

$$\mathrm{Rep}_k(\check{G}) \simeq \mathrm{Perv}_{G_\mathcal{O}}(\mathcal{G}r_G) \simeq \mathrm{Perv}_{IW,\mathbb{G}_m}(\mathcal{G}r_G) \tag{2}$$

where the first equivalence is the geometric Satake equivalence (due to [3]) and the second is the Iwahori–Whittaker realisation (due to [1]). Under these equivalences, the irreducible representation $L(\lambda)$ is sent to the IC sheaf $\mathcal{IC}^{IW}(\lambda+\rho)$ of the orbit closure $\overline{\mathcal{G}r_{G,\lambda+\rho}}$[1]. Moreover, the tilting module $T(\lambda)$ is sent to the parity sheaf $\mathcal{E}^{IW}(\lambda+\rho)$ (parity sheaves on the affine grassmannian are perverse thanks to the the Iwahori–Whittaker equivariance condition).

Now the linkage principle can be stated equivalently as

$$\text{if } W_{\text{aff},\ell}\lambda \neq W_{\text{aff},\ell}\mu \text{ then } \mathrm{Hom}_{\mathrm{Perv}_{IW,\mathbb{G}_m}(\mathcal{G}r_G,k)}(\mathcal{E}^{IW}(\lambda), \mathcal{E}^{IW}(\mu)) = 0 \tag{3}$$

[1] The Iwahori–Whittaker equivariance is very restrictive and only orbits corresponding to strongly dominant characters, i.e., belonging to $\rho + \check{X}^+$, can afford an IW local system!

where $W_{\mathrm{aff},\ell}$ is the ℓ-dilated affine Weyl group $W \ltimes \ell\check{X}$. Notice that the shift by ρ in $\mathcal{E}^{IW}$ and the dot action have cancelled out!

Such a vanishing of Ext groups between two sheaves is most satisfactorily explained geometrically when the sheaves have disjoint support. However, this is not the case here as the affine Grassmannian $\mathcal{G}r_G$ has very few connected components (and they are independent of ℓ!). A fix comes from considering fixed points under an extra symmetry: the loop rotation.

The loop rotation arises naturally in the theory of affine Kac–Moody groups as the action of a cocharacter of the maximal torus. In our context, it can be explained as follows: since $\mathbb{G}_m$ acts on $\mathrm{Spec}(\mathcal{K})$ by $tf(z) = f(tz)$ it induces a natural action on $G_{\mathcal{K}}$ and hence on $\mathcal{G}r_G$. The locus of fixed points under the loop rotation $\mathbb{G}_m$ on $\mathcal{G}r_G$ is a disjoint union over dominant characters $\lambda \in \check{X}^+$ of partial flag varieties. Since this is again independent of ℓ we need to look further.

Let $\mu_n \leq \mathbb{G}_m$ be the subgroup of n-th roots of unity. Bezrukavnikov made the following observation which was proved in [2]:

Proposition 1. *We have a decomposition into connected components*

$$\mathcal{G}r_G^{\mu_n} = \bigsqcup_{\gamma \in \check{X}/W_{\mathrm{aff},n}} \mathcal{G}r_{G,\gamma} \tag{4}$$

where each $\mathcal{G}r_{G,\gamma}$ is (the connected component of unity of) a partial affine flag variety of the form $G^\circ_{\mathcal{K}_n}/\mathcal{P}_\gamma$ where $\mathcal{K}_n = \mathbb{F}((z^n))$ and $\mathcal{P}_\gamma$ is a certain parahoric subgroup associated to γ.

The second ingredient is the Smith–Treumann localisation to fixed points of μ_ℓ on $\mathcal{G}r_G$. In the previous talk it was explained that if we have an action of $\mathbb{G}_m$ on a variety Y then we can define the (constructible) Smith–Treumann category as

$$\mathrm{Sm}_{\mathbb{G}_m}(Y^{\mu_\ell}, k) := \mathcal{D}^b_{\mathbb{G}_m}(Y^{\mu_\ell}, k) / \left\langle \begin{array}{c} \text{complexes whose restriction to } \mu_\ell \\ \text{are perfect } k[\mu_\ell]\text{-complexes} \end{array} \right\rangle.$$

It is a triangulated category which is 2-periodic, i.e., $\mathrm{id} \simeq [2]$.

Denote by $i\colon Y^{\mu_\ell} \to Y$ the inclusion of fixed points and by

$$Q\colon \mathcal{D}^b_{\mathbb{G}_m}(Y^{\mu_\ell}, k) \to \mathrm{Sm}_{\mathbb{G}_m}(Y^{\mu_\ell}, k)$$

the quotient functor. The main features of the Smith–Treuman theory [4] are captured by

Theorem 2 ([4])**.** *The natural map of functors $i^! \to i^*$ gives an isomorphism $Qi^! = Qi^*$. If we denote the resulting functor by*

$$i^{!*}\colon \mathcal{D}^b_{\mathbb{G}_m}(Y, k) \to \mathrm{Sm}_{\mathbb{G}_m}(Y^{\mu_\ell}, k)$$

then it commutes with all functors of the form f_*, $f_!$, f^*, $f^!$ for $\mathbb{G}_m$-equivariant maps f.

Since the Smith category is 2-periodic we can not make sense of perverse sheaves. However, as it was observed by Leslie–Lonnergan [5] one can make sense of parity sheaves. To this purpose the following important technical result (explained in the previous talk) is proved in [2]

Proposition 3. *In the Smith category of a point we have*

$$\mathrm{Ext}^*_{\mathrm{Sm}_{\mathbb{G}_m}(pt)}(\underline{k}, \underline{k}) = k[x, x^{-1}], \text{ where } \deg(x) = 2.$$

It is rather straightforward to make sense of the Smith–Treumann category in the Iwahori–Whittaker setting and therefore to talk about parity sheaves in $\mathrm{Sm}_{IW,\mathbb{G}_m}(\mathcal{G}r_G^{\mu_\ell}, k)$. We will denote by $\mathrm{Sm}^\sharp_{IW,\mathbb{G}_m}(\mathcal{G}r_G^{\mu_\ell}, k)$ the subcategory of even objects for the dimension pariversity[2]. We can now state the main result of [2] that was the subject of this talk:

Theorem 4. *The Smith–Treumann localisation functor*

$$i^{!*}\colon \mathcal{D}^b_{IW,\mathbb{G}_m}(\mathcal{G}r_G, k) \to \mathrm{Sm}_{IW,\mathbb{G}_m}(\mathcal{G}r_G^{\mu_\ell}, k)$$

induces an equivalence of categories between parity sheaves

$$i^{!*}\colon \mathrm{Parity}_{IW,\mathbb{G}_m}(\mathcal{G}r_G, k) \simeq \mathrm{Sm}^\sharp_{IW,\mathbb{G}_m}(\mathcal{G}r_G^{\mu_\ell}, k). \tag{5}$$

The decomposition into connected components (4) coupled with the above equivalence gives the decomposition of categories

$$\mathrm{Parity}_{IW,\mathbb{G}_m}(\mathcal{G}r_G, k) \simeq \bigoplus_{\gamma \in \check{X}/W_{\mathrm{aff},\ell}} \mathrm{Sm}^\sharp_{IW,\mathbb{G}_m}(\mathcal{G}r_{G,\gamma}, k)$$

which, according to the geometric reformulation (3), is precisely the linkage principle.

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[2]all the relevant orbits have constant parity in a connected component so the dimension pariversity and the constant pariversity give the same parity sheaves

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