§1.3 Vector equations

Def: ~~Vector~~ A vector is a matrix with one column (Column vector)

$\mathbb{R}^n$ is the set of $n \times 1$ matrices.

↖ Special properties: We add vectors and get another. We can scale vectors.

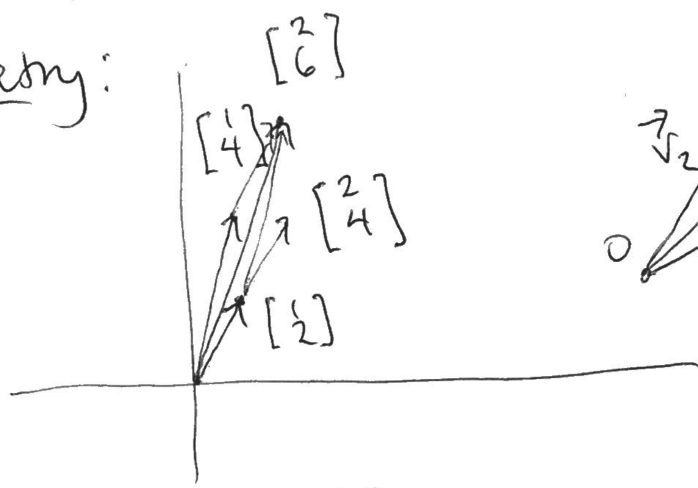
$\boxed{\mathbb{R}^2}$

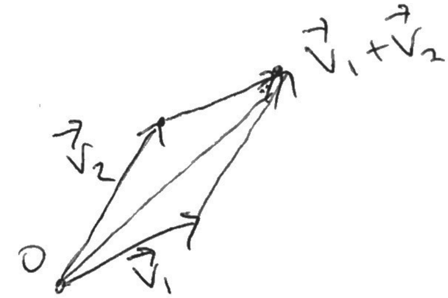
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Warning: There is no general way to multiply ~~two~~ vectors in $\mathbb{R}^n$ to get another vector.

Geometry:





Properties: $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ zero vector.

1) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ commutative law
2) $\vec{v} + \vec{0} = \vec{v} = \vec{0} + \vec{v}$ identity.
3) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) = \vec{u} + \vec{v} + \vec{w}$. assoc. law
4) $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$ distributive law.

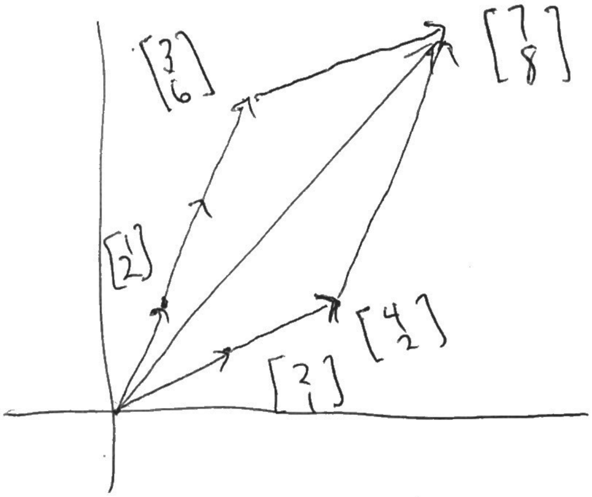
<u>Def</u>: A <u>linear combination</u> of $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ ← is an element of

is a vector of the form $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$, in

where $\underbrace{c_1, \dots, c_k}_{\text{weights of the linear combination}} \in \mathbb{R}$.

<u>Geometry of linear combinations</u>

$$2\begin{bmatrix}2\\1\end{bmatrix} + 3\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix} + \begin{bmatrix}3\\6\end{bmatrix} = \begin{bmatrix}7\\8\end{bmatrix}.$$



<u>Q</u>: Is $\begin{bmatrix}8\\22\end{bmatrix}$ a linear combination of $\begin{bmatrix}2\\1\end{bmatrix}$ & $\begin{bmatrix}1\\2\end{bmatrix}$?

If so, with which weights?

$$x\begin{bmatrix}2\\1\end{bmatrix} + y\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}8\\22\end{bmatrix} \quad \text{vector equation.}$$

$$= \begin{bmatrix}2x\\x\end{bmatrix} + \begin{bmatrix}y\\2y\end{bmatrix} = \begin{bmatrix}2x+y\\x+2y\end{bmatrix}$$

$$\begin{bmatrix}2x+y\\x+2y\end{bmatrix} = \begin{bmatrix}8\\22\end{bmatrix}.$$

$$\begin{cases} 2x + y = 8 \\ x + 2y = 22 \end{cases}$$

Is this system consistent?

$$\left[\begin{array}{cc|c} 2 & 1 & 8 \\ 1 & 2 & 22 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 - \frac{1}{2}R_1} \left[\begin{array}{cc|c} 2 & 1 & 8 \\ 0 & 3/2 & 18 \end{array}\right]$$ consistent

There is a unique solution since every column has a pivot.

$R_1 - \frac{2}{3}R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 12 \end{array}\right] \xleftarrow[R_2 \mapsto \frac{2}{3}R_2]{R_1 \mapsto \frac{1}{2}R_1} \left[\begin{array}{cc|c} 2 & 0 & -4 \\ 0 & 3/2 & 18 \end{array}\right]$$

weights are -2 and 12.

In fact, every (any) vector in $\mathbb{R}^2$ is a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with unique weights.

~~When is every~~

For which pairs of vectors $\vec{v}_1$ & $\vec{v}_2$ is every vector in $\mathbb{R}^2$ a linear combination of $\vec{v}_1$ & $\vec{v}_2$? (with ~~a~~ unique weights)?

Can't have $[0 \; \dots \; 0]$ in the coefficient matrix.

$[0 \;\; 0]$

coeff matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 - \frac{c}{a}R_1} \begin{bmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{bmatrix}$

$d \neq \frac{bc}{a}$ or $\boxed{ad - bc \neq 0}$.

Slopes $-\frac{a}{b}$, $-\frac{c}{d}$ $\quad ax + by = ? \quad cx + dy = ?$

Slopes are equal exactly when $ad - bc = 0$.

Slopes of vectors; $\frac{c}{a}$ & $\frac{d}{b}$, are equal when $ad - bc = 0$

Addendum: $0\vec{v} = \vec{0}$ for all $\vec{v} \in \mathbb{R}^n$

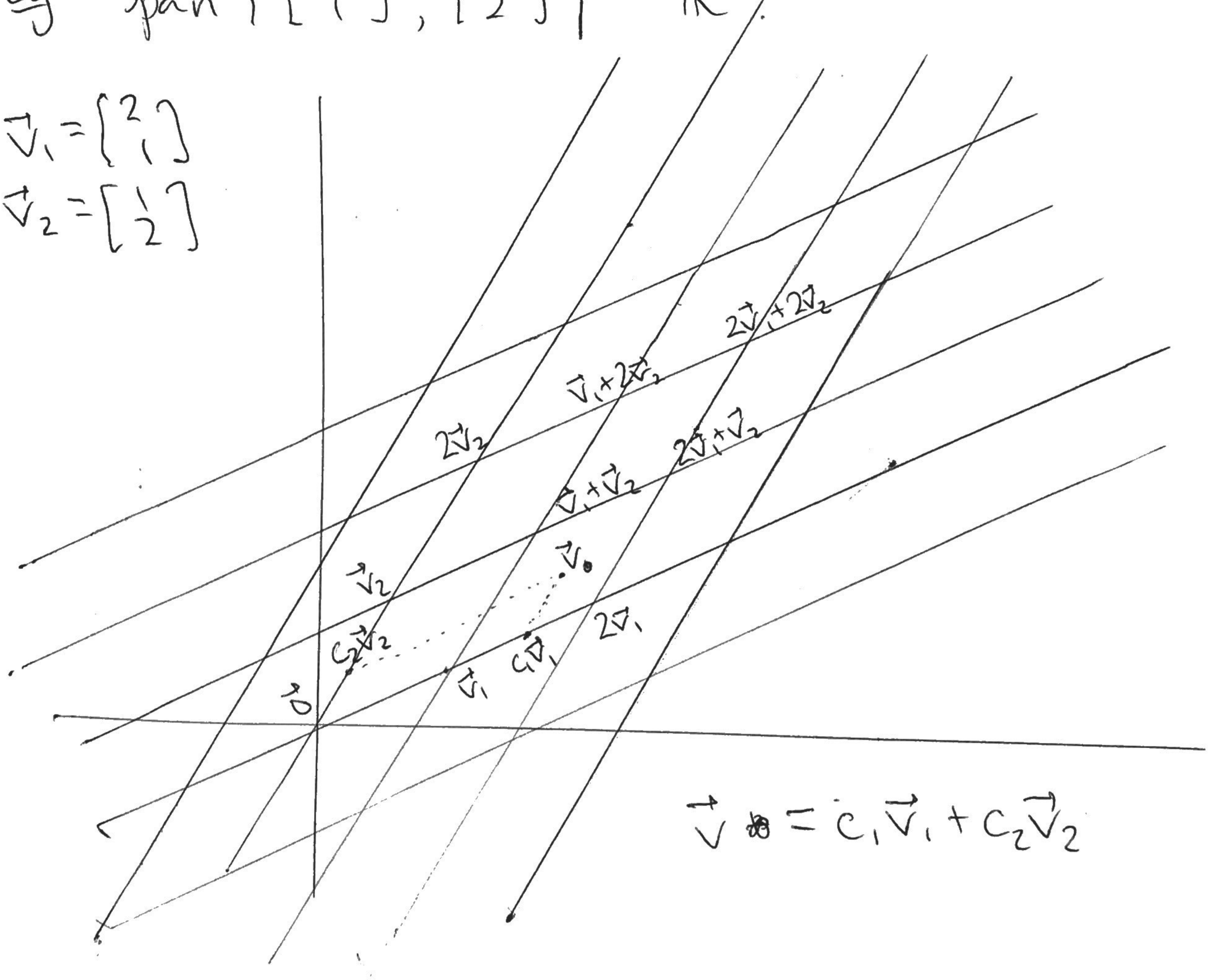
$-\vec{v}$ means $(-1)\vec{v}$

Def: The span of $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is the set of linear combinations of $\vec{v}_1, \ldots, \vec{v}_k$.

Eg: $\text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \mathbb{R}^2$

$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$