§1.4 The matrix equation $A\vec{x} = \vec{b}$.

matrix · vector = vector

$$x\begin{bmatrix}2\\1\end{bmatrix} + y\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}8\\22\end{bmatrix}$$

equiv to a system of lin eqns

$$x = -2,\ y = 12.$$

$$\begin{bmatrix}8\\22\end{bmatrix} \in \text{Span}\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$$

In fact, ↓ is $\mathbb{R}^2$.

<u>Notation</u>: matrix · vector

$$\begin{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}2\\1\end{bmatrix} + y\begin{bmatrix}1\\2\end{bmatrix}$$

$$\underset{m \times n}{\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{bmatrix}}\underset{n \times 1}{\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix}} = \underset{m \times 1}{x_1\vec{v}_1 + \cdots + x_n\vec{v}_n} \in \mathbb{R}^m.$$

$$i \rightarrow \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{e}_i \in \mathbb{R}^n. \quad \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \in \mathbb{R}^n.$$

I or I_n $(n \times n)$ identity matrix

$$\begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{e}_1 + \ldots + x_n\vec{e}_n$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \ldots + x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \ldots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

$I_n\vec{x} = \vec{x}$ for $\vec{x} \in \mathbb{R}^n$.

$$\begin{cases} x+2y+3z=1 \\ 4x+5y+6z=2 \\ 7x+8y+9z=3 \end{cases} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 3 \end{array}\right]$$

$$\begin{bmatrix} x+2y+3z \\ 4x+5y+6z \\ 7x+8y+9z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= x\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + z\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{\text{coeff matrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\text{vector from constants}}$$

System of linear eqns

$[A \mid \vec{b}]$

$\Updownarrow$ Solve for $\vec{x}$:

$A\vec{x} = \vec{b}$

Properties: 1) $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$

2) $A(c\vec{v}) = c(A\vec{v})$

(Convention: Capital letters = matrices

$\overrightarrow{\text{lower case}}$ = vector

lower case = real number (scalar))

Simple example:

$$\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1[a_1] + \cdots + x_n[a_n]$$

$$= [x_1 a_1] + \cdots + [x_n a_n]$$

$$= [x_1 a_1 + \cdots + x_n a_n].$$

$$\vec{b} = A\vec{x}$$

$$\begin{array}{c} \\ i \\ \\ \end{array} \begin{bmatrix} \\ [b_i] \\ \\ \end{bmatrix} = \begin{bmatrix} \\ [a_{i1}\ a_{i2} \cdots a_{in}] \\ \\ \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} \\ a_{i1} \\ \\ \end{bmatrix} + x_2 \begin{bmatrix} \\ a_{i2} \\ \\ \end{bmatrix} + \dots + x_n \begin{bmatrix} \\ a_{in} \\ \\ \end{bmatrix}$$

$$= \begin{bmatrix} \\ x_1 a_{i1} \\ \\ \end{bmatrix} + \begin{bmatrix} \\ x_2 a_{i2} \\ \\ \end{bmatrix} + \dots + \begin{bmatrix} \\ x_n a_{in} \\ \\ \end{bmatrix}$$

$$= \begin{bmatrix} \\ x_1 a_{i1} + \dots + x_n a_{in} \\ \\ \end{bmatrix}.$$

$$b_i = x_1 a_{i1} + \dots + x_n a_{in}.$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 12 \end{bmatrix} = \begin{bmatrix} -2 \cdot 2 + 12 \cdot 1 \\ -2 \cdot 1 + 12 \cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 22 \end{bmatrix}.$$

<u>Results</u>:

<u>Thm</u>: $A\vec{x} = \vec{b}$ has a solution if and only if (iff) $\vec{b}$ is a

linear combination of the columns of A.
(ie $\vec{b}$ is in the Span of the columns of A)

Thm: Let A be an $m \times n$ matrix. Then the following are equivalent (TFAE):

1) $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^m$. $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$
2) Each $\vec{b} \in \mathbb{R}^m$ is a linear combo of the columns of A.
3) The span of the columns of A is $\mathbb{R}^m$.
4) A has a pivot position in each row.

1), 2), & 3) are equivalent by definitions of $A\vec{x}$ and span.

Why are 1) & 4) equivalent?

If A has a pivot position in every row, REF(A) has no row of 0's and so system is always consistent.

If A does not have a pivot position in a row, one can find $\vec{b}$ so that the augmented matrix $[A|\vec{b}]$ corresponds to an <u>inconsistent</u> system.

Indeed row reduce $[A|\vec{b}]$

$$\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

treat as variables

$$[A \mid \vec{b}\,]$$

$$\Big\downarrow \text{ row reduce}$$

$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & * \end{array}\right]$$

$* = c_1 b_1 + \ldots + c_n b_n$ for some numbers c_i.

Choose b_i so that $* \neq 0$ and system is <u>inconsistent</u>.