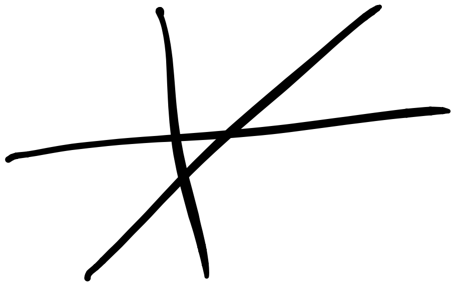
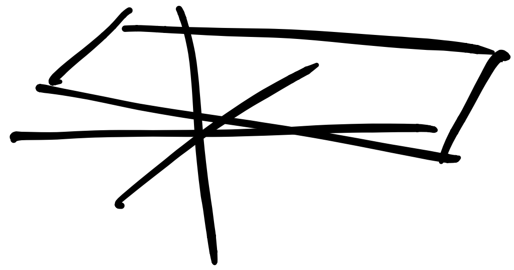
§ 1.5 Solution sets of linear equations





System of linear eqns

$[A \mid \vec{b}\,]$ $\qquad$ $A\vec{x} = \vec{b}$

Def: A system of linear eqns is <u>homogeneous</u> if all the constants are 0 i.e. it corresponds to a matrix equation of the form

$$A\vec{x} = \vec{0}.$$

A homogeneous system is <u>always</u> consistent.

A <u>nontrivial solution</u> [ie a solution with some variable x_i set to something nonzero] exists if and only if there is a free variable ie there is a column of A which has no pivot position.

Eg: $$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 0 \\ -3x_1 - 2x_2 + 4x_3 = 0 \\ 6x_1 + x_2 - 8x_3 = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right]$$

$\downarrow$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

pivot columns

free variable

There is a nontrivial solution.

$$\rightsquigarrow \left[\begin{array}{ccc|c} 3 & 0 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 - 4/3\, x_3 = 0 \\ x_2 \qquad\quad = 0 \end{cases}$$

$$\begin{cases} x_1 = 4/3\, x_3 \\ x_2 = 0 \\ x_3 \text{ is free} \end{cases}$$

$$\begin{cases} x_1 = 4/3\, t \\ x_2 = 0 \\ x_3 = t \end{cases}$$

parametric form

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \vec{x} = \vec{0}$$

$$\vec{x} = \begin{bmatrix} \frac{4}{3}t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

parametric vector form

<u>Homogeneous</u>: Any scalar multiple of a solution is again a solution.

Eg: $$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \vec{x} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

not homogeneous

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

consistent

$$\begin{cases} x_1 - 4/3\, x_3 = -1 \\ x_2 = 2 \end{cases}$$

$$x_1 = 4/3\, x_3 - 1$$

$$x_2 = 2$$

x_3 free

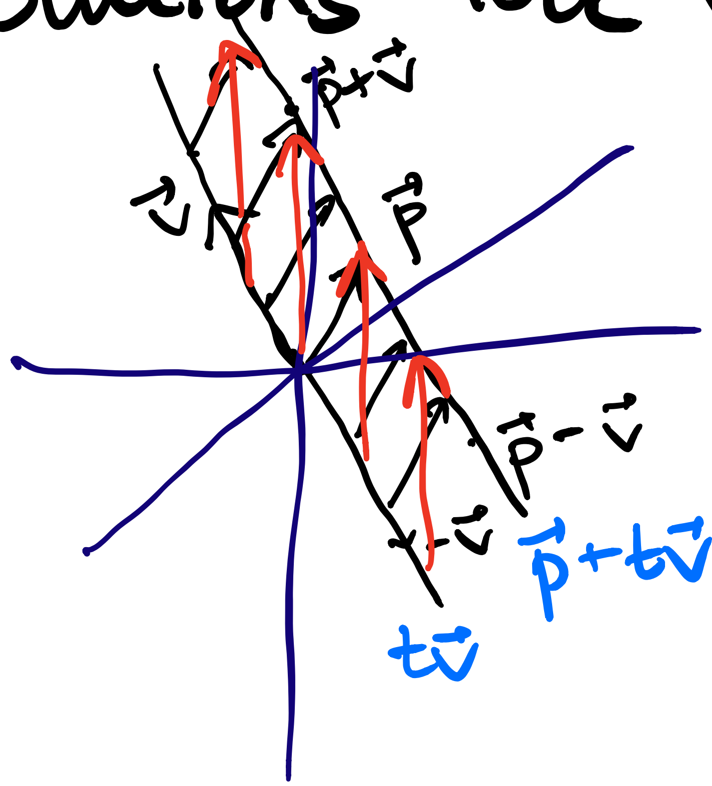
$$\begin{cases} x_1 = 4/3\, t - 1 \\ x_2 = 2 \\ x_3 = t \end{cases}$$

$$\vec{x} = \begin{bmatrix} 4/3\, t - 1 \\ 2 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4/3\, t \\ 0 \\ t \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

parametric vector form

The solutions look very similar.



$$\vec{v} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

If $\vec{v}$ and $\vec{w}$ are solutions to $A\vec{x} = \vec{b}$, then $\vec{v}-\vec{w}$ is a solution to $A\vec{x} = \vec{0}$.

<u>Reason</u>:
$$\begin{aligned} A(\vec{v}-\vec{w}) &= A\vec{v} - A\vec{w} \\ &= \vec{b} - \vec{b} \\ &= \vec{0}. \end{aligned}$$

If $\vec{p}$ is a solution to $A\vec{x}=\vec{b}$, all other solutions are of the form $\vec{p}+\vec{v}$ where $\vec{v}$ is a solution to $A\vec{x}=\vec{0}$.

$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is a solution to a homogeneous system. Is there a column in the coeff matrix w/o a pivot position?

Yes, since $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is a nontrivial solution.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + 2x_2 + 3x_3 = 0$$

$$x_1 = -2x_2 - 3x_3$$

x_2, x_3 free

$$x_1 = -2s - 3t$$

$$x_2 = s$$

$$x_3 = t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s-3t \\ s \\ t \end{bmatrix}$$

$$= \begin{bmatrix} -2s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -3t \\ 0 \\ t \end{bmatrix}$$

$$= s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$