2.9 Dimension and rank

Last time: Basis for a subspace H of $\mathbb{R}^n$.
← linear independent set which spans H.

Today: How to use a basis to define coordinates on a subspace and dimension of a subspace.

Key: Let $\{\vec{b}_1, \dots, \vec{b}_p\}$ be a basis of a subspace $H \subset \mathbb{R}^n$. ($p \leq n$ since if $p > n$ then any p vectors in $\mathbb{R}^n$ are linearly dependent.)

Then we can write any $\vec{v} \in H$ as a linear combo

$$\vec{v} = c_1\vec{b}_1 + \dots + c_p\vec{b}_p$$

in a unique way.

$H = \text{Span}\{\vec{b}_1, \dots, \vec{b}_p\}$ so $\vec{v}$ is a linear combo of $\vec{b}_1, \dots, \vec{b}_p$. Say $\vec{v} = c_1\vec{b}_1 + \dots + c_p\vec{b}_p$.

If $\vec{v} = d_1 b_1 + \dots + d_p \vec{b}_p$, then

$$c_1 = d_1,\ c_2 = d_2,\ \dots,\ c_p = d_p.$$

Why? $\vec{0} = (c_1 - d_1)\vec{b}_1 + \dots + (c_p - d_p)\vec{b}_p$.

$$\Rightarrow c_1 - d_1 = 0,\ \dots,\ c_p - d_p = 0.$$

$$\begin{bmatrix} 8 \\ 22 \end{bmatrix} = p\begin{bmatrix} 1 \\ 2 \end{bmatrix} + d\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$.

order matters

Say that $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ is a basis of $H \subset \mathbb{R}^n$.

If $\vec{v} = c_1\vec{b}_1 + \dots + c_p\vec{b}_p \in H$, then

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

$\vec{v}$ in the basis $\mathcal{B}$ is $\begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$.

Eg: $\begin{bmatrix} 8 \\ 22 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $\qquad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$

order matters

$$\begin{bmatrix} 8 \\ 22 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

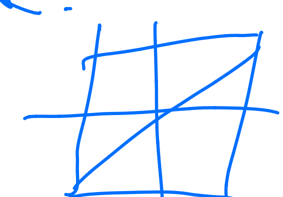
Eg: $\vec{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$.

$\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. $\qquad p = 2$, $n = 3$.

$\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is a basis for $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} = H$

$\leftarrow$ plane through $\vec{0}$ in $\mathbb{R}^3$.

Check that $\vec{x} \in H$ and find $[\vec{x}]_{\mathcal{B}}$.



$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 \rightsquigarrow \left[\begin{array}{cc|c} \vec{v}_1 & \vec{v}_2 & \vec{x} \end{array}\right].$$

$$\left[\begin{array}{cc|c} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \qquad c_1 = 2, c_2 = 3.$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

<u>Thm</u>: If $\{\vec{b}_1, \ldots, \vec{b}_p\}$ is a basis for $H \subset \mathbb{R}^n$, then any p linearly independent vectors in H span H and any p vectors which span H are linearly independent.

First consider $H = \mathbb{R}^n$. $\{\vec{e}_1, \ldots, \vec{e}_n\}$ standard basis

Any n linearly independent vectors in $\mathbb{R}^n$ span $\mathbb{R}^n$.

Any n vectors which span $\mathbb{R}^n$ are linearly independent.

Why?

$\{v_1, \ldots, \vec{v}_n\}$ is linearly independent $\Leftrightarrow$ $\mathrm{RREF}([\vec{v}_1 \cdots \vec{v}_n])$ has n pivots

$\Updownarrow$

iff $\mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_n\} = \mathbb{R}^n$. $\Leftrightarrow$

What if $H \neq \mathbb{R}^n$. Then H "behaves just like" $\mathbb{R}^p$.

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \vdots \end{bmatrix} \in \mathbb{R}^p.$$

$$[\vec{x}+\vec{y}]_B = [\vec{x}]_B + [\vec{y}]_B$$
$$[c\vec{x}]_B = c[\vec{x}]_B$$

$$[\cdot]_B : H \to \mathbb{R}^p.$$

<u>Corollary</u>: Any basis of H has p elements.

<u>Proof</u>: If $\{\vec{v}_1, \dots, \vec{v}_m\}$ is a basis

Any m linearly independent vectors

H span H.

If $m<p$, then $\vec{b}_1, \dots, \vec{b}_m$ span H.

If $m>p$, then $\vec{v}_1, \dots, \vec{v}_p$ span H. $\square$

If any (or all) bases of H has p elements, we say the dimension of $\cancel{H}$ is p.

$\text{Rank}(A) = \dim \text{Col}(A)$ $\quad$ A $m \times n$ matrix

basis comes from pivot columns of A.

$= \#$ pivot positions of A.

$\text{nullity}(A) = \dim \text{Nul}(A)$

Thm: # columns of A = Rank(A) + nullity(A)

pivot columns # free columns