§ 1.7 <u>Linear independence</u>

<u>Def</u>: A set $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly independent if the vector equation

$$x_1 \vec{v}_1 + \dots + x_p \vec{v}_p = \vec{0}$$

has only the trivial solution.

$$x_1 = x_2 = \dots = x_p = 0$$

is the <u>trivial solution</u>.

Def: The set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is linearly dependent if

$$c_1\vec{v}_1 + \ldots + c_p\vec{v}_p = 0 \text{ for}$$

some $c_i \in \mathbb{R}$ not all 0.

linear dependence relation

Eg: $\{[1]\}$ is linearly indep.

$$x[1] = \vec{0}$$

$$[x] = \vec{0}$$

$x = 0$ is the only

and trivial solution

Eg: Is $\{\vec{v}\}$ linearly independent?

$$x\vec{v} = \vec{0}$$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad x\vec{v} = \begin{bmatrix} xv_1 \\ \vdots \\ xv_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

If $v_i \neq 0$ for any i,

$$xv_i = 0 \Rightarrow x = 0,$$

and so $\{\vec{v}\}$ is linearly independent.

If $v_i = 0$ for all i,

then any element of $\mathbb{R}$ is a solution.

$\{\vec{0}\}$ is linearly dependent

<u>Fact</u>: $\{\vec{v}\}$ is linearly independent iff $\vec{v} \neq \vec{0}$.

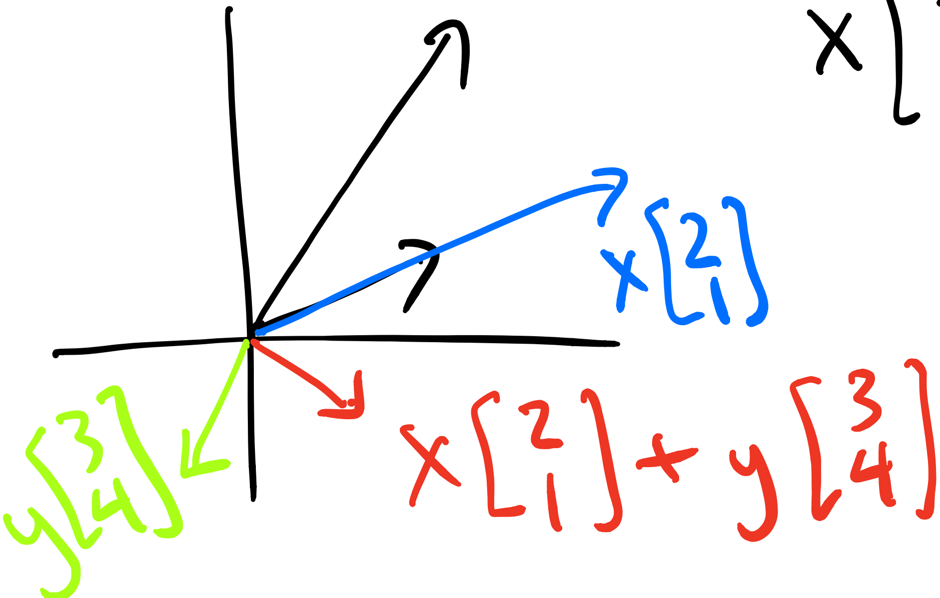
Eg: $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ linearly independent

$$x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 1 & 4 & 0 \end{array}\right] \rightsquigarrow \underbrace{\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 5/2 & 0 \end{array}\right]}_{\text{REF}}$$

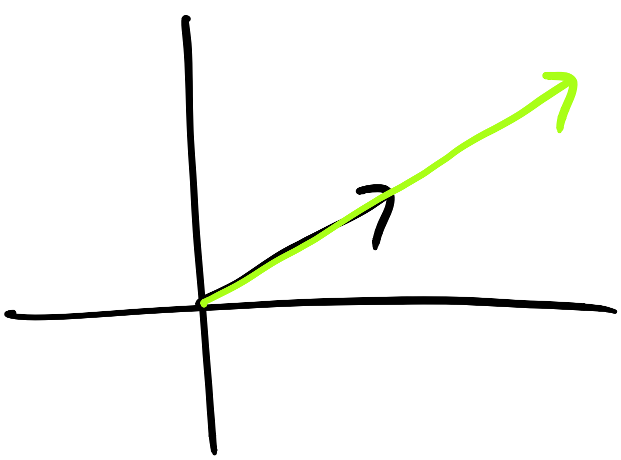
No free variables means no nontrivial solutions.

$$x\begin{bmatrix}2\\1\end{bmatrix}+y\begin{bmatrix}3\\4\end{bmatrix}=\vec{0}$$



Eg: $\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}4\\2\end{bmatrix}\right\}$ is linearly dependent

$$2\cdot\begin{bmatrix}2\\1\end{bmatrix}-\begin{bmatrix}4\\2\end{bmatrix}=\vec{0}$$



When is $\{\vec{v}, \vec{w}\}$ linearly independent?

$$c\vec{v} + d\vec{w} = \vec{0}$$ linear dependence relation

If $c \neq 0$, $\vec{v} = -\frac{d}{c}\vec{w}$.

If $d \neq 0$, $\vec{w} = -\frac{c}{d}\vec{v}$.

<u>Fact</u>: $\{\vec{v}, \vec{w}\}$ is linearly dependent iff $\vec{v} \in \text{Span}\{\vec{w}\}$ or $\vec{w} \in \text{Span}\{\vec{v}\}$.

Eg: If $\vec{w} = \vec{u} + \vec{v}$, is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent? No.

$\vec{u} + \vec{v} - \vec{w} = \vec{0}$ is a linear dependence relation.

Eg: If $\vec{w} \in \text{Span}\{\vec{u}, \vec{v}\}$, is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent? No.

$\vec{w} = c\vec{u} + d\vec{v}$ for some $c, d \in \mathbb{R}$.

$c\vec{u} + d\vec{v} - \vec{w} = \vec{0}$ is a linear dependence relation

Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

linearly independent? No.

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & -6 & 2 & 0 \end{array}\right]$$

$$\rightsquigarrow$$

$$\text{REF} \quad \left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable

Setting the free variable to be something nonzero, I get a nontrivial solution.

Thm: If $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly dependent, then $\{\vec{v}_1, \dots, \vec{v}_p, \vec{w}\}$ is linearly dependent as well.

Proof: A linear dependence relation $c_1\vec{v}_1 + \dots + c_p\vec{v}_p = \vec{0}$ stays a linear dependence relation $c_1\vec{v}_1 + \dots + c_p\vec{v}_p + 0\vec{w} = \vec{0}$.

Cor: If $\vec{0} \in S$, S is linearly dependent.

Thm: $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly dependent iff one vector is in the span of the others.

Pf: $(\Leftarrow)$ $\vec{v}_i = c_1\vec{v}_1 + \dots + c_p\vec{v}_p$

no $\vec{v}_i$ term

$$c_1\vec{v}_1 + \dots - \vec{v}_i + c_{i+1}\vec{v}_{i+1}$$
$$+ \dots + c_p\vec{v}_p = \vec{0}.$$

$(\Rightarrow)$ $c_1\vec{v}_1 + \dots + c_p\vec{v}_p$

$c_i \neq 0$ say

Solve $\vec{v}_i$.

Thm: If $p > n$, a set of p vectors in $\mathbb{R}^n$ is linearly dependent.

Pf: The augmented matrix one should solve is $n \times p$. Since $p > n$, there are more columns then rows, and so there must be a free variable

Thm: If $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly independent and $\vec{w} \notin \text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$, $\{\vec{v}_1, \dots, \vec{v}_p, \vec{w}\}$ is linearly independent.

Pf: Solve $x_1\vec{v}_1 + \dots + x_p\vec{v}_p + y\vec{w} = \vec{0}$

Is there a solution $(c_1, \ldots, c_p, d)$ with $d \neq 0$? No since otherwise

$$\vec{w} = \frac{1}{d}(c_1\vec{v}_1 + \ldots + c_p\vec{v}_p)$$
$$\in \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}.$$

How about $d = 0$?

Then $c_1\vec{v}_1 + \ldots + c_p\vec{v}_p = \vec{0}$, and $c_1 = c_2 = \ldots = c_p = 0$ since $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is linearly independent.

So there is <u>only</u> the trivial solution. $\square$