§1.8 Intro to linear transformations

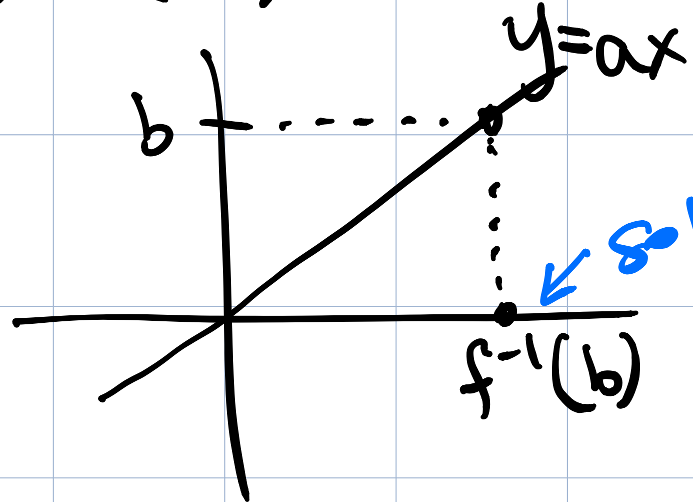
Systems of linear eqns $\rightsquigarrow$ $A\vec{x} = \vec{b}$

eg: $g(x) = \sin x$
$g(x) = x^2$

$ax = b$

$f(x) = ax$ $\quad g : \mathbb{R} \to \mathbb{R}$

$f^{-1}(b) =$ solution set to $ax = b$



$A \quad m \times n$ matrix

$f(\vec{x}) = A\vec{x}$ $\quad \vec{v} \in \mathbb{R}^n$
$f(\vec{v}) \in \mathbb{R}^m$.

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

transformation, map, function

$f^{-1}(\vec{b})$ = solution set to $A\vec{x} = \vec{b}$.

More complicated eg which will never come up in the class:

$f : \mathbb{R}^2 \to \mathbb{R}^3$

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \sin(xy) \\ x^2 - y \\ 2 \end{bmatrix}$$

Typical example for this class:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$T_A : \underbrace{\mathbb{R}^2}_{\text{domain of } T_A} \to \underbrace{\mathbb{R}^2}_{\text{codomain of } T_A}$$

codomain of T_A $\supseteq$ range of T_A = set of images of vectors

$$T_A(\vec{v}) = A\vec{v}$$ image of $\vec{v}$

$$T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x + 4y \end{bmatrix}$$

$f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$

$\text{range}(f) = \{a \in \mathbb{R} : a \geq 0\}$

$T_A(\vec{x}) = A\vec{x}$ is called a matrix transformation.

Properties:

$$T_A(\vec{v}+\vec{w}) = A(\vec{v}+\vec{w})$$
$$= A\vec{v} + A\vec{w}$$
$$= T_A(\vec{v}) + T_A(\vec{w})$$

$$T_A(c\vec{v}) = A(c\vec{v})$$
$$= c(A\vec{v})$$
$$= c\,T_A(\vec{v})$$

Def: A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear if

1) $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$

2) $T(c\vec{v}) = c\,T(\vec{v})$

Eg: Matrix transformations are linear transformations.

Eg: $f([x]) = [ax] \qquad f = T_{[a]}$

<u>Non-eg:</u> $g([x]) = [ax+b], \; b \neq 0$

$$g([x]) + g([y]) = [ax+b] + [ay+b]$$
$$= [a(x+y) + 2b]$$

X

$$g([x+y]) = [a(x+y)+b]$$

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$$

Is $\begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix} \in \text{range}(T_A)$? No

Does $A\vec{x} = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}$ have a solution? No

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & 4 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -10 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 8 \end{array}\right]$$

What is range(T_A)?

$\vec{b} \in \text{range}(T_A)$ iff $A\vec{x} = \vec{b}$ has a solution iff $\vec{b}$ is in the span of the columns of A.

$\text{range}(T_A) = \text{Span}\{\text{columns of } A\}$

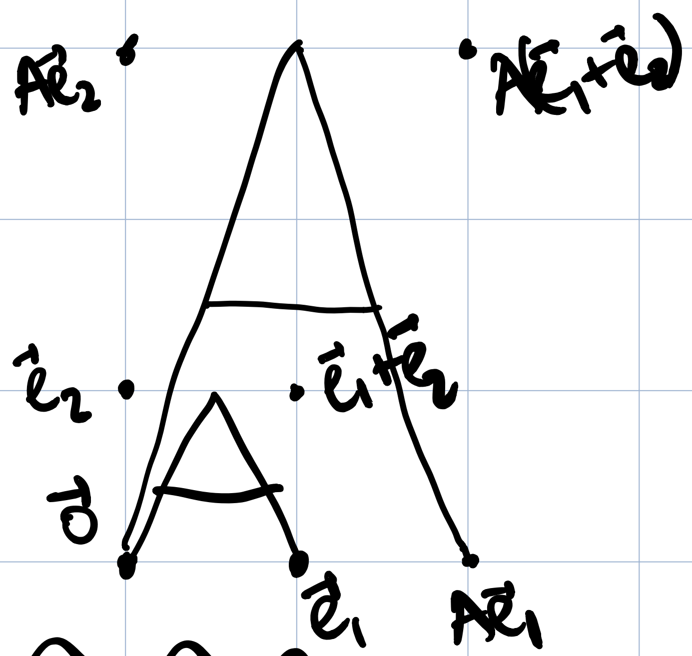
$= \text{Col}(A)$

column space of A

$$A = \begin{bmatrix} \begin{bmatrix} \vec{v}_1 \\ \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \vec{v}_n \\ \\ \\ \end{bmatrix} \end{bmatrix}$$

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad A\vec{e}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, A\vec{e}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$



$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

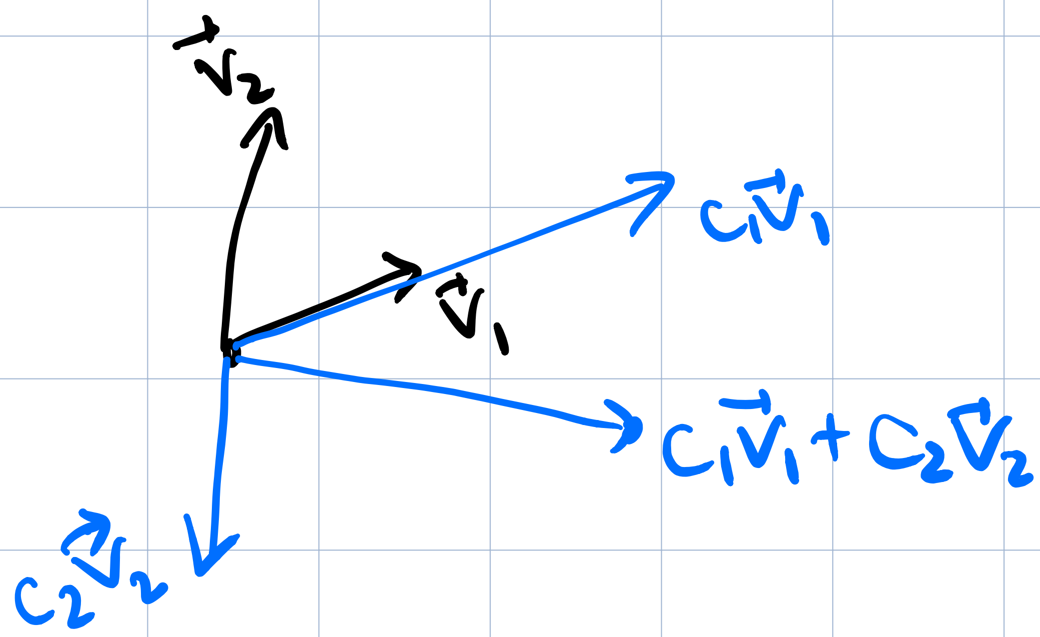
In response to questions...

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

$$\text{range}(T_A) = \text{Col}(A) = x\text{-axis}$$

$$\subset \mathbb{R}^2 = \text{codomain}(T_A)$$

$$\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 : c_1, c_2 \in \mathbb{R}\}$$



$$A = \begin{bmatrix} 1 & .5 \\ 0 & 1 \end{bmatrix}$$