HW 2-1

① We defined a function $f: \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd}. \end{cases}$$

Prove that this is a bijection.

② If $D \neq \emptyset$ is any set, let us say that a function $f: D \to \mathbb{R}$ is bounded (above) if the set

$$f(D) = \{f(x) : x \in D\} \subset \mathbb{R} \text{ is bounded (above).}$$

If f is bounded above we write $\sup\limits_D f$ for $\sup f(D) \in \mathbb{R}$.

Suppose $\varphi, \psi : D \to \mathbb{R}$ are bounded above. Prove that the function $\theta : D \to \mathbb{R}$ defined by

$$\theta(x) = \varphi(x) + \psi(x), \quad x \in D,$$

is also bounded above, and $\sup\limits_D \theta \leq \sup\limits_D \varphi + \sup\limits_D \psi$.

③ Can it occur that in ② $\sup\limits_D \theta < \sup\limits_D \varphi + \sup\limits_D \psi$?