HW 2-15

① Prove $\lim_{k\to\infty} \sqrt[k]{2} = 1$. Hint: Bernoulli's

② Suppose a sequence (y_n) is defined recursively by

$$y_1 = 4\ ; \quad y_{n+1} = \frac{1}{2}\left(y_n + \frac{3}{y_n}\right), \quad n \in \mathbb{N}.$$

Find (and, of course, prove) $\lim_{n\to\infty} y_n$.

③ Prove the binomial theorem: If $a, b \in \mathbb{R}$, $m \in \mathbb{N} \Rightarrow$

$$(a+b)^m = a^m + m a^{m-1} b + \frac{m(m-1)}{1\cdot 2} a^{m-2} b^2 + \ldots + \frac{m(m-1)\cdot\ldots\cdot 2\cdot 1}{1\cdot 2\cdot\ldots\cdot(m-1)m} b^m.$$

E.g., $(a+b)^3 = a^3 + 3a^2 b + \frac{3\cdot 2}{1\cdot 2} ab^2 + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3} b^3 = a^3 + 3a^2 b + 3ab^2 + b^3$.

HW 2-17

① (a) Are there unbounded sequences that have a convergent subsequence? (Recall: a sequence (c_k) is bounded if $\exists B \in \mathbb{R}$ s.t. $\forall k \in \mathbb{N}$ $|c_k| \leq B$; otherwise it is unbounded.)

(b) Are there such monotone sequences?

② Let $z_1 = 1$, $z_{n+1} = \frac{z_n^3 + 2}{5}$, $n \in \mathbb{N}$. Show that $\forall n$ $0 \leq z_n \leq 1$; (z_n) converges; its limit z satisfies $z^3 - 5z + 2 = 0$.

Hint: Induction!

HW 2-17 continued

(3) Prove that $\lim_{n\to\infty} a^n = 0$ if $|a|<1$, while $\lim_{n\to\infty} a^n = \infty$ if $a>1$.