HW 2-1

① We defined a function $f: \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Prove that this is a bijection.

② If $D \neq \emptyset$ is any set, let us say that a function $f: D \to \mathbb{R}$ is bounded (above) if the set $f(D) = \{f(x) : x \in D\} \subset \mathbb{R}$ is bounded (above).

If f is bounded above we write $\sup_D f$ for $\sup f(D) \in \mathbb{R}$.

Suppose $\varphi, \psi : D \to \mathbb{R}$ are bounded above. Prove that the function $\theta: D \to \mathbb{R}$ defined by

$$\theta(x) = \varphi(x) + \psi(x), \quad x \in D,$$

is also bounded above, and $\sup_D \theta \leq \sup_D \varphi + \sup_D \psi$.

③ Can it occur that in ② $\sup_D \theta < \sup_D \varphi + \sup_D \psi$?

2-3

① (a) If $n \in \mathbb{N}$, show that the set of meaningful English sentences containing n characters (letters, spaces, punctuation marks, math. symbols) is finite.

(b) Show that the set of all meaningful English sentences is countable.

In these problems, let's be liberal with the meaning of "meaningful". E.g., "A rose is a rose is a rose." should count as meaningful.

HW 2-3 continued

(2) Show $|[0,1]| = |[0,2]|$.

(3) Show $|(0,1)| = |(0,\infty)|$.