Thomson, Bruckner²: Elementary real analysis

$\times$	a_0	a_1	a_2	a_3	a_4	a_5	$\dots$
b_0	a_0b_0	a_1b_0	a_2b_0	a_3b_0	a_4b_0	a_5b_0	$\dots$
b_1	a_0b_1	a_1b_1	a_2b_1	a_3b_1	a_4b_1	a_5b_1	$\dots$
b_2	a_0b_2	a_1b_2	a_2b_2	a_3b_2	a_4b_2	a_5b_2	$\dots$
b_3	a_0b_3	a_1b_3	a_2b_3	a_3b_3	a_4b_3	a_5b_3	$\dots$
b_4	a_0b_4	a_1b_4	a_2b_4	a_3b_4	a_4b_4	a_5b_4	$\dots$
b_5	a_0b_5	a_1b_5	a_2b_5	a_3b_5	a_4b_5	a_5b_5	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	

Figure 3.3. The product of the two series $\sum_0^\infty a_k$ and $\sum_0^\infty b_k$.

(which with $x = 1$ is the same question we just asked)? The now obvious answer is

$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2$$
$$+(a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + \dots.$$

Notice that this method of grouping the terms corresponds to summing along diagonals of the rectangle in Figure 3.3.

This is the source of the following definition.

Definition 3.51 The series

$$\sum_{k=0}^{\infty} c_k$$

is called the *formal product* of the two series

$$\sum_{k=0}^{\infty} a_k \text{ and } \sum_{k=0}^{\infty} b_k$$

provided that

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}.$$

Our main goal now is to determine if this "formal" product is in any way a genuine product; that is, if

$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} a_k \times \sum_{k=0}^{\infty} b_k.$$

The reason we expect this might be the case is that the series $\sum_{k=0}^{\infty} c_k$ contains all the terms in the expansion of

$$(a_0 + a_1 + a_2 + a_3 + \dots)(b_0 + b_1 + b_2 + b_3 + \dots).$$

A good reason for caution, however, is that the series $\sum_{k=0}^{\infty} c_k$ contains these terms only in a particular arrangement and we know that series can be sensitive to rearrangement.

3.8.1 Products of Absolutely Convergent Series

It is a general rule in the study of series that absolutely convergent series permit the best theorems. We can rearrange such series freely as we have seen already in Section 3.7.1. Now we show that we can form products of such series. We shall have to be much more cautious about forming products of nonabsolutely convergent series.

Theorem 3.52 (Cauchy) *Suppose that* $\sum_{k=0}^{\infty} c_k$ *is the formal product of two absolutely convergent series*

$$\sum_{k=0}^{\infty} a_k \text{ and } \sum_{k=0}^{\infty} b_k.$$

Then $\sum_{k=0}^{\infty} c_k$ *converges absolutely too and*

$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} a_k \times \sum_{k=0}^{\infty} b_k.$$

Proof We write

$$A = \sum_{k=0}^{\infty} a_k, \; A' = \sum_{k=0}^{\infty} |a_k|, \; A_n = \sum_{k=0}^{n} a_k,$$

$$B = \sum_{k=0}^{\infty} b_k, \; B' = \sum_{k=0}^{\infty} |b_k|, \; \text{and} \; B_n = \sum_{k=0}^{n} b_k.$$

By definition

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}$$

and so

$$\sum_{k=0}^{N} |c_k| \le \sum_{k=0}^{N} \sum_{i=0}^{k} |a_i| \cdot |b_{k-i}| \le \left(\sum_{i=0}^{N} |a_i|\right)\left(\sum_{i=0}^{N} |b_i|\right) \le A'B'.$$

Since the latter two series converge, this provides an upper bound $A'B'$ for the sequence of partial sums $\sum_{k=1}^{N} |c_k|$ and hence the series $\sum_{k=0}^{\infty} c_k$ converges absolutely.

Let us recall that the formal product of the two series is just a particular rearrangement of the terms $a_i b_j$ taken over all $i \ge 0$, $j \ge 0$. Consider any arrangement of these terms. This must form an absolutely convergent series by the same argument as before since $A'B'$ will be an upper bound for the partial sums of the absolute values $|a_i b_j|$. Thus all rearrangements will converge to the same value by Theorem 3.48.

We can rearrange the terms $a_i b_j$ taken over all $i \geq 0$, $j \geq 0$ in the following convenient way "by squares." Arrange always so that the first $(m+1)^2$ $(m = 0, 1, 2, \dots)$ terms add up to $A_m B_m$. For example, one such arrangement starts off

$$a_0 b_0 + a_1 b_0 + a_0 b_1 + a_1 b_1 + a_2 b_0 + a_2 b_1 + a_0 b_2 + a_1 b_2 + a_2 b_2 + \dots.$$

(A picture helps considerably to see the pattern needed.) We know this arrangement converges and we know it must converge to

$$\lim_{m \to \infty} A_m B_m = AB.$$

In particular, the series $\sum_{k=0}^{\infty} c_k$ which is just another arrangement, converges to the same number AB as required. ■

It is possible to improve this theorem to allow one (but not both) of the series to converge nonabsolutely. The conclusion is that the product then converges (perhaps nonabsolutely), but different methods of proof will be needed. As usual, nonabsolutely convergent series are much more fragile, and the free and easy moving about of the terms in this proof is not allowed.

3.8.2 Products of Nonabsolutely Convergent Series

Let us give a famous example, due to Cauchy, of a pair of convergent series whose product diverges. We know that the alternating series

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{\sqrt{k+1}}$$

is convergent, but not absolutely convergent since the related absolute series is a p-harmonic series with $p = \frac{1}{2}$.

Let

$$\sum_{k=0}^{\infty} c_k$$

be the formal product of this series with itself. By definition the term c_k is given by

$$(-1)^k \left[\frac{1}{\sqrt{1 \cdot (k+1)}} + \frac{1}{\sqrt{2 \cdot (k)}} + \frac{1}{\sqrt{3 \cdot (k-1)}} \dots + \frac{1}{\sqrt{(k+1) \cdot 1}} \right].$$