Rigorous introduction of natural numbers

In class we said: the natural numbers are $1, 1+1, 1+1+1, 1+1+1+1, \ldots,$ and so on.

This "and so on" is a little vague. Is there a better way to explain what natural numbers are? Yes, there is, in fact more than one way; but all are a little more involved than the "and so on" definition. Here is one way to introduce natural numbers.

First, a general construction. Suppose we are given an arbitrary set $I \neq \emptyset$, and with each $i \in I$ is associated a set A_i. The intersection and the union of the family A_i, $i \in I$, of sets is

<u>Def</u>: $$\bigcap_{i \in I} A_i = \{x : \forall i \;\; x \in A_i\},$$

$$\bigcup_{i \in I} A_i = \{x : \exists i \text{ s.t. } x \in A_i\}.$$

E.g., if $I = \{1,2\}$, then $\bigcap_{i \in I} A_i = \{x : x \in A_1 \text{ and } x \in A_2\}$
$= A_1 \cap A_2$.

Rigorous, continued

Now consider subsets $S \subset \mathbb{R}$ with this property:

(P) $\quad 1 \in S$; if $x \in S \Rightarrow x+1 \in S$.

E.g. $S = \mathbb{R}$ and $S = \mathbb{P}$ have this property; $S = \{0, 1/2, 1\}$ and $S = \emptyset$ don't.

List all such sets as S_i, $i \in I$. Thus each S_i has (P), and if some $S \subset \mathbb{R}$ has (P), then it is one of the S_i.

Def: $\mathbb{N} = \bigcap_{i \in I} S_i$.

Prop: (i) $\mathbb{N}$ has property (P)

(ii) If $S \subset \mathbb{R}$ has property (P), then $\mathbb{N} \subset S$.

Pf: Construct the proof for yourself.

If $\mathbb{N}$ is introduced as above, then the "Principle of mathematical induction" becomes a theorem, that can be proved. The proof is not particularly difficult.