We shall now prove the existence of nth roots of positive reals. This proof will show how the difficulty pointed out in the Introduction (irrationality of $\sqrt{2}$) can be handled in R.

1.21 Theorem *For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$.*

This number y is written $\sqrt[n]{x}$ or $x^{1/n}$.

Proof That there is at most one such y is clear, since $0 < y_1 < y_2$ implies $y_1^n < y_2^n$.

Let E be the set consisting of all positive real numbers t such that $t^n < x$.

If $t = x/(1 + x)$ then $0 \le t < 1$. Hence $t^n \le t < x$. Thus $t \in E$, and E is not empty.

If $t > 1 + x$ then $t^n \ge t > x$, so that $t \notin E$. Thus $1 + x$ is an upper bound of E.

Hence Theorem 1.19 implies the existence of

$$y = \sup E.$$

To prove that $y^n = x$ we will show that each of the inequalities $y^n < x$ and $y^n > x$ leads to a contradiction.

The identity $b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$ yields the inequality

$$b^n - a^n < (b - a)nb^{n-1}$$

when $0 < a < b$.

Assume $y^n < x$. Choose h so that $0 < h < 1$ and

$$h < \frac{x - y^n}{n(y + 1)^{n-1}}.$$

Put $a = y$, $b = y + h$. Then

$$(y + h)^n - y^n < hn(y + h)^{n-1} < hn(y + 1)^{n-1} < x - y^n.$$

Thus $(y + h)^n < x$, and $y + h \in E$. Since $y + h > y$, this contradicts the fact that y is an upper bound of E.

Assume $y^n > x$. Put

$$k = \frac{y^n - x}{ny^{n-1}}.$$

Then $0 < k < y$. If $t \ge y - k$, we conclude that

$$y^n - t^n \le y^n - (y - k)^n < kny^{n-1} = y^n - x.$$

Thus $t^n > x$, and $t \notin E$. It follows that $y - k$ is an upper bound of E.

But $y - k < y$, which contradicts the fact that y is the *least* upper boun of E.

Hence $y^n = x$, and the proof is complete.

Corollary *If a and b are positive real numbers and n is a positive integer, the*

$$(ab)^{1/n} = a^{1/n}b^{1/n}.$$

Proof Put $\alpha = a^{1/n}$, $\beta = b^{1/n}$. Then

$$ab = \alpha^n\beta^n = (\alpha\beta)^n,$$

since multiplication is commutative. [Axiom (M2) in Definition 1.12 The uniqueness assertion of Theorem 1.21 shows therefore that

$$(ab)^{1/n} = \alpha\beta = a^{1/n}b^{1/n}.$$

1.22 Decimals We conclude this section by pointing out the relation betwee real numbers and decimals.

Let $x > 0$ be real. Let n_0 be the largest integer such that $n_0 \le x$. (Note tha the existence of n_0 depends on the archimedean property of R.) Having chose $n_0, n_1, \ldots, n_{k-1}$, let n_k be the largest integer such that

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \le x.$$

Let E be the set of these numbers

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \qquad (k = 0, 1, 2, \ldots). \tag{5}$$

Then $x = \sup E$. The decimal expansion of x is

$$n_0 \cdot n_1 n_2 n_3 \cdots. \tag{6}$$

Conversely, for any infinite decimal (6) the set E of numbers (5) is bounde above, and (6) is the decimal expansion of $\sup E$.

Since we shall never use decimals, we do not enter into a detaile discussion.

THE EXTENDED REAL NUMBER SYSTEM

1.23 Definition The extended real number system consists of the real field R and two symbols, $+\infty$ and $-\infty$. We preserve the original order in R, and define

$$-\infty < x < +\infty$$

for every $x \in R$.