HW 1/9/23

① On $M = \mathbb{R}^2$ let $\rho(x,y) = |x_1 - y_1| + |x_2 - y_2|$, $x = (x_1, x_2)$, $y = (y_1, y_2)$

Show this is a metric, and sketch some balls.

② Can you define a metric on $\mathbb{R}$ such that

(a) Every subset $G \subset \mathbb{R}$ is open?

(b) Every sequence in $\mathbb{R}$ converges (in this metric)?

③ If in (M, ρ) we have two convergent sequences $x_n \to x$ and $y_n \to y$ $(n \to \infty)$, prove that

$$\lim_{n \to \infty} \rho(x_n, y_n) = \rho(x, y).$$

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① Prove that if A is any set, $F_\alpha \subset M$ are closed $\forall \alpha \in A$, then $\bigcap_{\alpha \in A} F_\alpha$ is also closed.

② The diadic distance on rational numbers.

Given a rational $\frac{a}{b} \in \mathbb{Q}$, choose a $k \in \mathbb{Z}$ such that $\frac{a}{b} = \frac{\alpha}{2^k \beta}$, where α, β are odd. (E.g. if $\frac{a}{b} = \frac{3}{5}$, then $k = 0$; if $\frac{a}{b} = \frac{4}{5}$, then $k = -2$, since $\frac{4}{5} = \frac{1}{2^{-2} \cdot 5}$.)

The diadic norm of $\frac{a}{b}$ is $\left|\frac{a}{b}\right|_2 = 2^k$.

Prove (a) $\left|\frac{a}{b} + \frac{c}{d}\right|_2 \le \max\left(\left|\frac{a}{b}\right|_2, \left|\frac{c}{d}\right|_2\right)$

(b) $\rho\left(\frac{a}{b}, \frac{c}{d}\right) = \left|\frac{a}{b} - \frac{c}{d}\right|_2$ defines a metric on ~~R~~ $\mathbb{Q}$.