HW 1-20-23

1) Let (M,ρ) be an arbitrary metric space. Prove that if a Cauchy sequence $x_n \in M$ has a convergent subsequence, then the whole sequence converges.

2) Let M consist of absolutely convergent series of real numbers. That is, $(x_1, x_2, \dots, x_n, \dots) \in M$ means: each $x_n \in \mathbb{R}$, and $\sum_{n=1}^{\infty} |x_n| < \infty$.

(a) Show that if $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots) \in M$ $\Rightarrow \sum_{n=1}^{\infty} |x_n - y_n|$ converges.

(b) Denoting the sum by $\rho(x,y)$, prove that ρ is a metric on M.

3) Prove that (M,ρ) of Problem 2 is a complete metric space.

1-23-23

1) Let (M,ρ) be a complete metric space and $f: M \to M$ a function s.t. $\rho(f(x), f(y)) \le \rho(x,y)/2 \ \forall x,y \in M$. With an arbitrary $x \in M$ consider $x, f(x), f(f(x)), f(f(f(x))), \dots, \underbrace{f(f(\dots f(x))\dots)}_{f^{[n]}(x)}, \dots$

Prove that the sequence $f^{[n]}(x)$ converges, and its limit z satisfies $f(z) = z$.

2) Prove that a closed subset of a compact metric space is always compact.

3) Let (M,ρ) be any metric space, $K \subset M$ compact, $F \subset M$ closed. Prove that if $K \cap F = \emptyset$, then $\exists r > 0$ s.t. $\forall k \in K, \forall f \in F$ $\rho(k,f) > r$.