HW 2-17-23

① Suppose $A \subset [0,\infty)$, $B \subset (-\infty, 0]$. Prove $m^*(A \cup B) = m^*(A) + m^*(B)$.

② Suppose $E \subset \mathbb{R}$ has the following property: for any interval I $m^*(E \cap I) \leq |I|/2$. Prove that E has measure 0.

③ Suppose $f : [0,1] \to \mathbb{R}$ is differentiable and f' is continuous. Let $C = \{x : f'(x) = 0\}$ denote the set of critical points. Show that $f(C) \subset \mathbb{R}$, the set of critical values, has measure 0.

2-20-23

① Prove that outer measure is translation invariant. That is, suppose $A \subset \mathbb{R}$, $d \in \mathbb{R}$, and $A + d = \{a + d : a \in A\}$; then $m^*(A+d) = m^*(A)$.

② If $f : \mathbb{R} \to \mathbb{R}$ is continuous, prove that its graph $G_f = \{(x, f(x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2$ is a closed set ($\mathbb{R}^2$ endowed with the Euclidean metric). Is the converse true? If $G_f \subset \mathbb{R}^2$ is closed $\overset{?}{\Rightarrow} f$ is continuous?

③ Suppose $g : [0,1) \to [0,1)$ is defined by

$$g(x) = \begin{cases} 2x & \text{if } x < 1/2 \\ 2x - 1 & \text{if } x \geq 1/2 \end{cases}.$$

If $A \subset [0,1]$, show $m^*(A) = m^*(g^{-1}A)$.