HW 2-24-23

① In previous problem #1, show that it may not be possible to choose A to be closed, not just F_σ.

② Let $q_n \in (0,1)$, $n \in \mathbb{N}$. Modify Cantor's original construction of his set so that in the n'th step we remove the middle q_n'th of each remaining interval; starting with $[0,1]$. (So in Cantor's construction $q_n = 1/3 \ \forall n$.) What measure does the (q_n)-Cantor set thus constructed have? Can it have positive measure for some choice of q_n?

③ Let $r_1, r_2, \ldots \in \mathbb{R}$ be a sequence, and $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \sum_{r_n < x} 2^{-n}$. Show that for any $A \subset \mathbb{R}$

$m^*_g(A) = \ldots = \sum_{n: r_n \in A} 2^{-n}$. What A will be measurable?

HW 2-27-23

① Express the characteristic functions $\chi_{A \cap B}$, $\chi_{A \cup B}$ and $\chi_{\Omega \setminus A}$ through χ_A, χ_B; here $A, B \subset \Omega$.

② If $f_n \in L(\Omega, \mathcal{A}, \mu)$, prove that $\{x: \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is measurable.

③ Let $\varphi \in L(\Omega, \mathcal{A}, \mu)$ and $\mathcal{E} = \{S \subset \mathbb{R} : \varphi^{-1}(S) \in \mathcal{A}\}$. Prove that $\mathcal{E}$ is a σ-algebra. Conclude from this that the pre-image $\varphi^{-1}(B)$ of any Borel set $B \subset \mathbb{R}$ is in $\mathcal{A}$.

φ^{-1}