HW 3-10-23

(1) Let $\lfloor x \rfloor$ denote the largest integer $\le x$, where $x \in \mathbb{R}$. (So $\lfloor 3 \rfloor = 3$, $\lfloor \pi \rfloor = 3$, $\lfloor -\sqrt{2} \rfloor = \lfloor -1.414\ldots \rfloor = -2$.) Define a function $A(x) = \frac{(-1)^{\lfloor x \rfloor}}{\lfloor x \rfloor}$, $x \ge 1$. Show that the improper Riemann integral $\int_1^\infty A$ exists, but A is not Lebesgue integrable on $[1, \infty)$.

(2) Suppose $f_n \in L(\Omega, \mathcal{A}, \mu)$ and $\lim_{n\to\infty} \int f_n^2 = 0$. Prove that there is a subsequence $n_1 < n_2 < \ldots < n_k < \ldots$ s.t. $f_{n_k} \to 0$ a.e. $(k \to \infty)$.

(3) In the readme folder, read the Egorov file, about Egorov's and Lusin's theorems. It is copied from the book "Introduction to real functions ..." by Sz. Nagy, and formulates E's theorem only for subsets of $\mathbb{R}$; but do check that the proof works for any measure space and E a measurable subset of finite measure.

HW 3-20-23

(1) Let $a, b \in \mathbb{R}$, $a < b$ and $E \subset [a, b]$. Prove that the characteristic function χ_E is Riemann integrable over $[a, b]$ if and only if ∂E has measure 0.

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(2) If $g \in \mathcal{L}^1[a,b]$ (with respect to Lebesgue measure), prove that $G(x) = \int_a^x g$ is a continuous function of $x \in [a,b]$.

(3) Given a measure space $(\Omega, \mathcal{A}, \mu)$ and $E_1, \dots, E_n \in \mathcal{A}$ of finite measure, prove that the matrix $\left(\mu(E_i \cap E_j)\right)_{i,j=1}^{n}$ is positive semidefinite, i.e.,

$$\sum_{i,j=1}^{n} \mu(E_i \cap E_j)\, \xi_i \xi_j \geq 0 \quad \text{whenever } \xi_i \in \mathbb{R},\ i=1,\dots,n.$$