HW 3-20-23

② If $g \in \mathcal{L}^1[a,b]$ (with respect to Lebesgue measure), prove that $G(x) = \int_a^x g$ is a continuous function of $x \in [a,b]$.

③ Given a measure space $(\Omega, \mathcal{A}, \mu)$ and $E_1, \dots, E_n \in \mathcal{A}$ of finite measure, prove that the matrix $\left(\mu(E_i \cap E_j)\right)_{i,j=1}^{n}$ is positive semidefinite, i.e.,

$$\sum_{i,j=1}^{n} \mu(E_i \cap E_j)\, \xi_i \xi_j \geq 0 \quad \text{whenever } \xi_i \in \mathbb{R},\ i = 1, \dots, n.$$

3-22-23

① Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $\mu(\Omega) < \infty$, $f_n \in L$ $(n \in \mathbb{N})$. If $f_n \to \infty$ a.e., and $\varepsilon > 0$, prove that $\exists E \in \mathcal{A}$ with $\mu(E) < \varepsilon$ such that $f_n \to \infty$ uniformly on $\Omega \setminus E$. But before you prove it, define what "$f_n \to \infty$ uniformly" means.

② Suppose $I_j \subset \mathbb{R}$ are intervals, $\sum_{j=1}^{n} |I_j| < \infty$, and

$$g(x) = \sum_{j=1}^{\infty} \left| I_j \cap (-\infty, x] \right|, \quad x \in \mathbb{R}.$$

Prove that $g: \mathbb{R} \to \mathbb{R}$ is an increasing function, and if x is contained in infinitely many I_j's, then g is not differentiable at x.