HW 3-24-23

① Evaluate asymptotically $\int_0^\infty e^{-\lambda x} f(x)\,dx$ as $\lambda \to \infty$, in the spirit of an example in class. Here $f \in C[0,\infty)$ is assumed to be bounded.

② A measure space $(\Omega, \mathcal{A}, \mu)$ is σ-finite if $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ with $\mu(\Omega_n) < \infty$, $\forall n \in \mathbb{N}$. Suppose $\varphi_1, \varphi_2 \geq 0$ are measurable functions on such Ω and

$$\int_E \varphi_1\, d\mu = \int_E \varphi_2\, d\mu \quad \text{for } \forall E \in \mathcal{A}.$$

Prove that $\varphi_1 = \varphi_2$ a.e.

③ If $u: \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, prove that $\exists\, v: \mathbb{R} \to \mathbb{R}$ Borel measurable s.t. $u = v$ a.e.

3-27-23

① Suppose I_n, $n \in \mathbb{N}$ are intervals, $\sum_{n=1}^{\infty} |I_n| < \infty$, and $g(x) = \sum_{n=1}^{\infty} |I_n \cap (-\infty, x]|$. Prove that $g(x)$ increases, and is not differentiable at x if x is contained in infinitely many I_n's.

Show that given an arbitrary set $E \subset \mathbb{R}$ of measure 0, there is an increasing function $h: \mathbb{R} \to \mathbb{R}$ that is not differentiable there.

3-27-23

(2) Construct an increasing function $\mathbb{R} \to \mathbb{R}$, discontinuous at every rational point.

(3) Prove that any $F \in \mathcal{L}^1(\Omega, \mathcal{A}, \mu)$ is uniformly integrable in the following sense: $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $\mu(E) < \delta$, then $\left| \int_E F \right| < \varepsilon$.