HW 3-31-23

① If $h \in \mathcal{L}^1(\Omega, \mathcal{A}, \mu)$, show that there is a sequence $g_n : \Omega \to \mathbb{R}$ of simple functions s.t.

$$\int_\Omega |g_n - h| \, d\mu \longrightarrow 0 \qquad (n \to \infty).$$

② A function g on an interval $[a,b]$ is piecewise ~~bounded~~ constant if there are $a = t_0 < t_1 < \dots < t_m = b$ such that the restrictions $g|_{(t_{j-1}, t_j)}$ are constant $\forall j = 1, \dots, m$.

If $f \in \mathcal{L}^1[a,b]$ and $\varepsilon > 0$, show that there is a piecewise constant $g : [a,b] \to \mathbb{R}$ s.t.

$$\int_a^b |f - g| < \varepsilon.$$

③ Suppose $\varphi_j \in L(\Omega, \mathcal{A}, \mu)$ are uniformly bounded, $j \in \mathbb{N}$: $\sup\{|\varphi_j(x)| : j \in \mathbb{N}, x \in \Omega\} < \infty$. If $\int_\Omega \varphi_i \varphi_j = 0$ whenever $i \neq j$, prove that the averages

$$\psi_n = \frac{\varphi_1 + \varphi_2 + \dots + \varphi_{n^2}}{n^2}$$

tend to 0 a.e.