HW 3-3-23

① Suppose $\psi \in L(\Omega, \mathcal{A}, \mu)$ is nonnegative. For $E \in \mathcal{A}$ define $\nu(E) = \int \psi \chi_E \in [0, \infty]$. (Here χ_E is the characteristic function of E.) Prove that $(\Omega, \mathcal{A}, \nu)$ is a measure space.

② Let $\mathcal{A}$ be a σ-algebra of subsets of Ω and $A_n \in \mathcal{A}$, $n \in \mathbb{N}$. Prove that $\{x \in \Omega : \exists$ infinitely many n such that $x \in A_n\} \in \mathcal{A}$.

③ Suppose $f \in L(\Omega, \mathcal{A}, \mu)$ and $\int f < \infty$. Prove that $f < \infty$ a.e.

HW 3-6-23

① Let ψ and ν be as in problem ① above. Suppose $g \in \mathcal{L}^1(\Omega, \mathcal{A}, \nu)$, and prove

$$\int_\Omega g \, d\nu = \int_\Omega g \psi \, d\mu .$$

Hint: Use the definition of integral.

② Consider the measure space $(\mathbb{N}, \{\text{all subsets}\}, \mu)$, where μ is counting measure. A function $f : \mathbb{N} \to \mathbb{R}$ is the same as a sequence $a_1 = f(1)$, $a_2 = f(2), \ldots, a_k = f(k), \ldots$. What functions are measurable? Which are integrable?

③ If $F_n \in L(\Omega, \mathcal{A}, \mu)$, $n \in \mathbb{N}$, $F_n \geq 0$ and $\sum_1^\infty \int F_n < \infty$, prove that $\sum_1^\infty F_n$ converges a.e., and $F_n \to 0$ $(n \to \infty)$ a.e.