HW 4-12-23

① Suppose $\mu(\Omega)<\infty$ and $f\in L^\infty(\Omega,\mathcal{A},\mu)$. Prove that $\lim_{p\to\infty}\|f\|_p=\|f\|_\infty$.

② A family of functions $\mathcal{F}\subset L^p(\Omega,\mathcal{A},\mu)$ is uniformly p-integrable if $\forall\varepsilon>0\ \exists\delta>0$ s.t.

$$\int_E |f|^p<\varepsilon \quad \text{when } \mu(E)<\delta \text{ and } f\in\mathcal{F}.$$

(Here $1\le p<\infty$.) Suppose $K\subset L^p(\Omega,\mathcal{A},\mu)$ is a compact subset. Prove that it is uniformly p-integrable.

HW 4-14-23

① If $f\in C_0(\mathbb{R}^n)$ and $f_t(x)=f(x-t)$, prove that the function

$$\mathbb{R}^n\ni t\longmapsto f_t\in L^p(\mathbb{R}^n)$$

is continuous. (Here $1\le p\le\infty$.)

② Give example of more than countably many functions $g_\alpha\in L^\infty[0,1]$ s.t. $\|g_\alpha-g_\beta\|_\infty\ge 1$ when $\alpha\ne\beta$. Show hence that $L^\infty[0,1]$ is not separable.

③ Suppose $\mu(\Omega)<\infty$ and $f\in L^{p_0}(\Omega,\mathcal{A},\mu)$. Prove that the function

$$[1,p_0]\ni p\longmapsto \|f\|_p\in\mathbb{R}$$

is continuous.