HW 4-21-23

① Prove the case $p=1$ of the Lemma in class:
Suppose $\Psi \in L^1(\Omega, \mathcal{A}, \mu)$ has the property that, with some $c \in \mathbb{R}$

$$\left| \int \Psi f \right| \leq c \|f\|_1 .$$

Then Ψ is essentially bounded, and even $\|\Psi\|_\infty \leq c$.

② Construct a measurable function for the square $[0,1] \times [0,1] \subset \mathbb{R}^2$ for which

$$\forall x \quad \int_0^1 f(x,y)\,dy = 0 , \quad \forall y \quad \int_0^1 f(x,y)\,dx = 1 .$$

Can you arrange that f is integrable?

③ A sequence of measurable functions g_k on a measure space $(\Omega, \mathcal{A}, \mu)$ converges to a measurable $g: \Omega \to (-\infty, \infty)$ <u>in measure</u> if $\forall \varepsilon > 0$

$$\lim_{k \to \infty} \mu \{ x \in \Omega : |g_k(x) - g(x)| > \varepsilon \} = 0 .$$

If $1 \leq p \leq \infty$ and $h_k \to h$ in $L^p(\Omega, \mathcal{A}, \mu)$, prove that $h_k \to h$ in measure as well.