Structure theorem of open sets in $\mathbb{R}$, ρ = Euclidean

Any $G \subset \mathbb{R}$ open is the union of a family of disjoint open intervals. Family is finite or countable.

could be half lines, or $\mathbb{R}$ itself

<u>Pf</u>: See Royden, Chapter 2, Prop. 8

(Not needed.)

Notation $A \subset B$ means:

$\forall x \in A$ is also $x \in B$.

($A = B$ is possible)

($A \subseteq B$ is also used.) (But not in this class.)

Most important metric spaces:

$\mathbb{R}$, $\mathbb{R}^k$, with Euclidean ρ,

$[a,b] \subset \mathbb{R}$

$$\rho(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\dots+(x_k-y_k)^2}$$

$x=(x_1,\dots,x_k)$, $y=(y_1,\dots,y_k)$.

3 important properties of metric spaces arising in connection with these.

I: <u>Cauchy's</u> thm (criterium)

$x_n \in \mathbb{R}$, $n=1,2,\dots$

Then x_n is convergent $\Longleftrightarrow$

$\forall \varepsilon > 0\ \exists N$ s.t.

$|x_n - x_m| < \varepsilon$ when $n,m > N$.

Prop: In any (M,ρ) if $x_n \in M$ converge $\Longrightarrow$

$$\underbrace{\forall \varepsilon>0\ \exists N \text{ s.t. } \rho(x_n,x_m)<\varepsilon \text{ when } n,m>N.}$$

If (x_n) has this property, we say: (x_n) is a Cauchy sequence

Pf: Let $x=\lim\limits_{n\to\infty} x_n \Longrightarrow$

$\rho(x,x_n)\to 0$ as $n\to\infty$.

Let $\varepsilon>0$. Choose $N\in\mathbb{N}$ s.t.

$\rho(x,x_n)<\varepsilon/2$ when $n>N$.

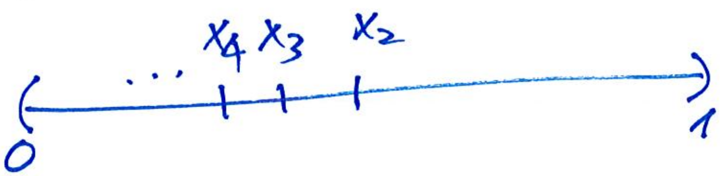
If now $n,m>N \Longrightarrow$

$$\rho(x_n,x_m)\le \underbrace{\rho(x_n,x)}_{<\varepsilon/2}+\underbrace{\rho(x,x_m)}_{<\varepsilon/2}<\varepsilon.$$

q.e.d.

Converse true in $\mathbb{R}$, $\mathbb{R}^k$, not in general.

Ex: $M=(a,b)=(0,1)$, $\rho=$ Euclidean



$x_n=\dfrac{1}{n}$, $n\ge 2$.

This is Cauchy, e.g. since when $n,m>N \Longrightarrow$

$$\rho(x_n,x_m)=\left|\frac{1}{n}-\frac{1}{m}\right|\le\frac{1}{n}+\frac{1}{m}<\frac{1}{N}+\frac{1}{N}=\frac{2}{N}$$

Given $\varepsilon>0$, choose N s.t. $\frac{2}{N}<\varepsilon$, that will do.

But: x_n is not convergent.

Suppose $x_n \to x \in M$

$$0 \leftarrow \rho(x_n, x) = \left|\frac{1}{n} - x\right| \to |x|$$

$\Rightarrow x = 0$. But $0 \notin M$.

↯ (contradiction)

<u>Def:</u> (M, ρ) is a <u>complete</u> metric space if any Cauchy sequence $x_n \in M$ converges.

In a complete metric space knowing that certain problems have arbitrarily good approximate solutions, implies they have exact solutions.

How to decide if (M, ρ) is complete?

From earlier classes: $\mathbb{R}$, $\mathbb{R}^k$, $[a, b]$ are complete; some others can be proved to be complete; certain operations with complete metric spaces give rise to cplete spaces.

E.g.:

Prop: Suppose (M, ρ) is cplete, $E \subset M$, ρ_E restriction of ρ to E. (E, ρ_E) is cplete $\Longleftrightarrow$ E is closed.

Pf: $(\Longleftarrow)$ Let $x_n \in E$ form Cauchy sequence, i.e.:

$\forall \varepsilon > 0 \ \exists N$ s.t.

$\rho_E(x_n, x_m) < \varepsilon$ when $n, m > N$.

$$\rho_E(x_n, x_m) = \rho(x_n, x_m).$$

So: $x_n \in M$ form Cauchy seq.

Hence x_n has limit in M $= x \in M$.

As E is closed $\Rightarrow x \in E$, and

$\rho_E(x, x_n) = \rho(x, x_n) \to 0 \quad (n \to \infty)$,

So: $x_n \to x$ in (E, ρ_E).

$(\Rightarrow)$: To show E is closed, take $y_n \in E$, $y_n \to y \in M$. Need to check $y \in E$.

So (y_n) Cauchy sequence in $M \Rightarrow$ " " in E

$\xRightarrow{E \text{ cplete}}$ y_n converges in E as well; $y_n \to z \in E$.

$$\underset{\substack{\vee\!\!\mid \\ 0}}{\rho(y, z)} \le \rho(y, y_n) + \rho_E(y_n, z) \to 0 + 0 \quad n \to \infty$$

$\Rightarrow \rho(y, z) = 0 \Rightarrow y = z \in E$. ✓

(For this direction, did not use M is complete)

II. When $M=[a,b]$:

Bolzano-Weierstrass selection thm:

Any sequence $x_n \in [a,b]$ has a convergent subsequence

$$x_{n_1}, x_{n_2}, \ldots \longrightarrow x \in [a,b].$$

Recall: $x_{n_1}, x_{n_2}, \ldots, x_{n_j}, \ldots$ is subsequence means:

$$n_1 < n_2 < \ldots < n_j < \ldots$$

Important in metric spaces, "compactness".

Definition: $K \subset M$ is compact means:

Given any family $G_\alpha \subset M$ $(\alpha \in A)$ of open sets s.t. $K \subset \bigcup_{\alpha \in A} G_\alpha$

$\Longrightarrow \exists$ finitely many $\alpha_1, \ldots, \alpha_m \in A$ s.t. $K \subset G_{\alpha_1} \cup \ldots \cup G_{\alpha_m}$.

"Any open cover of K has a finite subcover".

When $K = M$ is compact, we say: M is compact metric space.

Misconception: "~~K is compact if can be covered by finitely many open sets~~" WRONG

(Any set can be covered by a single open set, $G = M$.)