Sz. Nagy: Introduction to real functions

Dictionary: Point set = metric space
Neighborhood $k(P)$ of P = a ball in the metric space about P.

2.2.2. Necessary conditions for a sequence of continuous functions to have a continuous limit function

We have seen in the preceding paragraph that the uniformity or at least quasi-uniformity of the convergence of a sequence of continuous functions ensures that the limit function be also continuous. We shall prove now that in certain cases these sufficient conditions are also *necessary.*

Dini's Theorem. *Suppose $f_1, f_2, \ldots, f_n, \ldots$ are continuous functions on a compact point set H, and the sequence $\{f_n\}$ converges on H monotonically (increasing or decreasing) to a limit function f, which is also continuous on H. Then the convergence $f_n \to f$ is uniform on H.*

Proof. It suffices to consider monotonically decreasing sequences, and we may also assume that $f(P) = 0$ since the general case can be reduced to this one by replacing each f_n by $f_n - f$ (which is also continuous). Thus we consider the case

$$f_1(P) \geq f_2(P) \geq \ldots \geq f_n(P) \geq \ldots; \quad f_n(P) \to 0.$$

Let there be given an $\varepsilon > 0$. By virtue of the convergence to 0 there corresponds to each point $P \in H$ an index $m = m(P)$ for which

$$f_m(P) < \varepsilon.$$

By virtue of the continuity of the function f_m there is a neighborhood $k(P)$ of P such that for any point Q of H lying in $k(P)$ we also have

$$f_m(Q) < \varepsilon.$$

Thus we have assigned to each point $P \in H$ an index $m(P)$ and a neighborhood $k(P)$; the system of all these neighborhoods obviously covers the set H. By Borel's theorem, this system contains a finite subsystem $\{k(P_1), k(P_2), \ldots, k(P_r)\}$ which also covers the set H; thus any point $Q \in H$ lies in at least one of these neighborhoods. If Q lies in $k(P_i)$ then the inequality

$$f_n(Q) < \varepsilon$$

holds for the index $n = m(P_i)$, and by the monotonicity of the sequence it holds therefore for all the indices $n \geq m(P_i)$ too. Putting

$$n_0 = \max\{m(P_1), \ldots, m(P_r)\},$$

this inequality certainly holds for $n \geq n_0$. Now, the index n_0 does not depend on the choice of the point Q (it depends only on ε). This proves that the convergence of the sequence $\{f_n\}$ is uniform on H.

If we omit the condition of monotonicity, the theorem fails. However, we can prove

Arzelà's Theorem. *Suppose $\{f_n\}$ is a sequence of continuous functions on the compact point set H, converging on H to a continuous limit function f. Then the convergence is necessarily quasi-uniform.*

Proof. We may again suppose that $f(P) = 0$. In order to prove the quasi-uniformity of the convergence, take $\varepsilon > 0$ and