proof). Denoting by $e_n(x)$ and $e^*(x)$ the characteristic functions of the sets E_n and E^*, we have $e_n(x) \to e^*(x)$ and

$$\varphi_n(x)\,[1 - e_n(x)] \to f^*(x) = \begin{cases} f(x)\,[1 - e^*(x)] = f(x) & \text{for } x \in \complement E^* \\ 0 \text{ for } x \in E^*. \end{cases}$$

The step functions $\varphi_n(x)$ belong to the first Baire class, as do the characteristic functions of the open sets E_n. It follows that the products $\varphi_n(x)\,[1 - e_n(x)]$ also belong (at most) to the first Baire class, hence $f^*(x)$ belongs (at most) to the second Baire class. But obviously $f^*(x)$ differs from $f(x)$ only on the set of measure zero E^*.

5.3.7. The theorems of Egoroff and Lusin.

The concepts of measurable functions, and of pointwise convergence almost everywhere of measurable functions, are of course far reaching generalizations of the concepts of continuous functions and of uniform convergence. However, these generalizations are strongly related to the concepts of classical analysis, which they generalize, as shown by the following theorems which are due to Egoroff (1911) and Lusin (1912).

EGOROFF'S THEOREM. *If the sequence of measurable functions $f_n(x)$, defined on the interval (a, b), converges pointwise on a set $E \subset (a, b)$, of finite measure, we can remove from E a subset of arbitrary small measure in such a way that on the rest of E the convergence is uniform.*

PROOF. Denote the characteristic function of the set E by $e(x)$. The functions $g_n(x) = e(x) \cdot f_n(x)$ are measurable: we have $g_n(x) = f_n(x)$ on E, and $g_n(x) = 0$ elsewhere. The functions $g_n(x)$ converge *everywhere*, and their limit $g(x)$ is also measurable.

Let $\{\varepsilon_r\}$ be a decreasing sequence of positive numbers, tending to 0. Since the functions in question are measurable, the sets

$$F_{n,r} = \{x:\ |\,g(x) - g_n(x)\,| < \varepsilon_r\}\ (n, r = 1, 2, \ldots)$$

are measurable, as are the sets

$$E_{n,r} = E \cap F_{n,r} \cap F_{n+1,r} \cap F_{n+2,r} \cap \ldots \quad (n, r = 1, 2, \ldots);$$

$E_{n,r}$ consists of those points x of E for which

$$| g(x) - g_\nu(x) | < \varepsilon_r \text{ if } \nu \geq n$$

Clearly,

and
$$E_{1,r} \subset E_{2,r} \subset \ldots \subset E_{n,r} \subset E_{n+1,r} \subset \ldots \tag{31}$$

$$\bigcup_{n=1}^{\infty} E_{n,r} = E\,, \tag{32}$$

since the fact that the sequence $g_n(x)$ converges to $g(x)$ everywhere implies that every point x in E belongs to $E_{n,r}$ for n sufficiently large. From (31) and (32) it follows that

$$m(E_{n,r}) \to m(E) \text{ when } n \to \infty \text{ (cf. } \mathbf{5.3.2}\text{)}\,,$$

hence to each r we aan attach an $n(r)$ such that

$$m(E - E_{n(r),\,r}) < \frac{\delta}{2^r}$$

δ being an arbitrarily small, given positive number. Now put

$$E' = \bigcap_{r=1}^{\infty} E_{n(r),r}\,;$$

then we have

$$E - E' = \bigcup_{r=1}^{\infty} (E - E_{n(r),r})$$

and consequently

$$m(E - E') \leq \sum_{r=1}^{\infty} m(E - E_{n(r),r}) < \sum_{r=1}^{\infty} \frac{\delta}{2^r} = \delta\,.$$

Since $E' \subset E_{n(r),r}$, we have at each point x of the set E'

$$| g(x) - g_\nu(x) | < \varepsilon_r \text{ for } \nu \geq n(r)\,.$$

Thus the sequence $\{g_n(x)\}$, hence also $\{f_n(x)\}$, converges uniformly on the set $E' \subset E$, and $m(E - E') < \delta$.

Thus the theorem is proved.

From Egoroff's theorem we shall deduce

LUSIN'S THEOREM. *If $f(x)$ is a measurable function on the interval (a, b) then we can remove from (a, b) a set of arbitrarily small measure such that the function $f(x)$ when restricted to the rest of (a, b), is continuous.*

PROOF. It obviously suffices to consider the case where (a, b) is *finite*. Since $f(x)$ is measurable, there exists a sequence $\{\varphi_n(x)\}$ of step functions tending almost everywhere to $f(x)$. The points where the sequence does not converge to $f(x)$ and the points of discontinuity of the functions $\varphi_n(x)$ form a set E_1 of measure 0. Given a $\delta > 0$ we can find, by Egoroff's theorem, a subset E_2 of the set $(a, b) - E_1$ such that $m(E_2) < \delta$ and the convergence $\varphi_n(x) \to f(x)$ is uniform on $(a, b) - E$, where $E = E_1 \cup E_2$, $m(E) < \delta$. On the set $(a, b) - E$ all the functions $\varphi_n(x)$ are continuous, thus we can apply the theorem that the uniform limit of continuous functions is also continuous (cf. **2.2.1**). Hence $f(x)$ will be continuous if we restrict its definition to the set $(a, b) - E$. The theorem is proved.

Remark. Since every set of measure less than δ can be covered by a system of open intervals of total length less than δ, we can find an open set E' which encloses the set E in question and for which $m(E') < \delta$. Since $f(x)$ is continuous when restricted to $(a, b) - E'$, it is easy to construct a function $g(x)$ which is continuous on the entire interval (a, b) and which coincides with $f(x)$ on $(a, b) - E'$ (particular case, for linear sets, of Tietze's theorem, cf. **2.3.3**). Thus we can also formulate Lusin's theorem in the following form:

If $f(x)$ is a measurable function on the interval (a, b), then for every $\delta > 0$ we can find a function $g(x)$, continuous at every point of (a, b) and differing from $f(x)$ at most on a set of measure $< \delta$.