MA 266 Lecture 14

Section 3.2 Solutions of Linear Homogeneous Equations; Wronskian

Terminologies

In this section, we study the structure of solutions of second order linear differential equation.

Let p and q be continuous functions on an open interval I. For any function ϕ that is twice differentiable on I, we define the __________________________ L by

Note that $L[\phi]$ is also a function on I. For example, let $p(t) = t^2$, $q(t) = 1 + t$, and $\phi(t) = \sin(3t)$, then

The second order homogeneous linear equation can be written as

associated with initial conditions:

The theoretical result of existence and uniqueness of solution is stated in the theorem.

Theorem 3.2.1 (Existence and Uniqueness) Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y'_0.$$

Example 1. *Find the longest interval in which the solution of the following initial value problem is certain to exist.*

$$\begin{cases} (t^2 - 3t)y'' + ty' - (t+3)y = 0, \\ y(1) = 2, \quad y'(1) = 1. \end{cases}$$

Example 2. *Find the solution of the initial value problem*

$$\begin{cases} y'' + p(t)y' + q(t)y = 0, \\ y(0) = 0, \quad y'(0) = 0. \end{cases}$$

Theorem (Principle of Superposition) If y_1 and y_2 are two solutions of the differential equation

$$L[y] = y'' + py' + qy = 0,$$

then

Proof.

Remark. Beginning with only two solutions,

Question: Are there any solution of different form?

- The initial conditions require that
- The determinant of the matrix is
- If $W \neq 0$, then

The determinant W is called the ________________ of the solution y_1 and y_2.

Usually, we write it as ________________.

Theorem Suppose y_1 and y_2 are two solutions of

$$L[y] = y'' + py' + qy = 0,$$

Then

Remark.

- The theorem states that

- In this case, we say the expression

- The solutions y_1 and y_2 are said to form

Example 3. *Show that $y_1(t) = t^{1/2}$, and $y_2(t) = t^{-1}$ form a fundamental set of solution of*

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0.$$