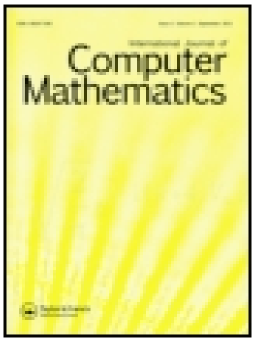

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A high-order difference scheme for the fractional sub-diffusion equation

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ABSTRACT

Based on the Lubich's high-order operators, a second-order temporal finite-difference method is considered for the fractional sub-diffusion equation. It has been proved that the finite-difference scheme is unconditionally stable and convergent in L^2 norm by the energy method in both one- and two-dimensional cases. The rate of convergence is order of two in temporal direction under the initial value satisfying some suitable conditions. Some numerical examples are given to confirm the theoretical results.

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1. Introduction

During the past several decades, the study of fractional differential equations has attracted many scholars' attention. The most important reason is that the fractional differential equations with fractional operators, which enjoy non-local connectivity, can be more accurate than the classical differential equations in the description of physical and chemical processes with non-local connectivity. For more relevant references and books, readers can refer to [2,14,22,24–26]. Like integer order differential equations, the exact solutions of the fractional differential equations are not available under most circumstances. Even if their solutions can be found, they are usually in the forms of series, which are difficult to evaluate. So the numerical investigation of the fractional differential equations has been a popular topic in recent years.

In this work, we are concerned with a high-order finite-difference method for the fractional sub-diffusion equation as follows:

$$\partial_t u(x,t) = (K_1\, {}_0D_t^{\alpha} + K_2\, {}_0D_t^{\beta})\partial_x^2 u(x,t) + p(x,t), \quad 0 < x < L, \quad 0 < t \le T, \tag{1}$$

$$u(0,t) = \varphi_0(t), \quad u(L,t) = \varphi_1(t), \quad 0 < t \le T, \tag{2}$$

$$u(x,0) = \psi(x), \quad 0 \le x \le L, \tag{3}$$

where $0 < \alpha, \beta < 1$, and K_1, K_2 are positive constants. Without loss of generality, we assume the initial condition $\psi(x) = 0$. If $\psi(x) \ne 0$, one may consider the equation for $v(x,t) = u(x,t) - \psi(x)$ instead. Therefore, the solution can be continuously extended to be zero for $t < 0$.

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There are several definitions of fractional derivatives, among which the two most commonly used ones are the Caputo and Riemann–Louville derivatives. Under some regularity assumptions, they can be converted to each other. According to this, fractional diffusion equations with Riemann–Louville derivatives are equivalent to ones in the Caputo form. For more details, see [9,17].

For the time-fractional diffusion equations, there have been a lot of numerical work. Langlands and Henry [15] considered and analysed the L1 scheme for the approximation of the fractional order time derivative. Sun and Wu [28] constructed a difference scheme with L1 approximation for the fractional diffusion equations and proved that the convergence rate is order of $2-\alpha$ in time, where α is the order of the time fractional derivative. Yuste [32] and Yuste and Acedo [33] proposed the explicit and weighted averaged difference schemes based on the Grünwald–Letnikov approximation and analysed the obtained schemes using the Von Neumann method. Zhuang *et al.* [36,37] introduced a new way to solve linear and nonlinear sub-diffusion equations. They first integrated the original differential equation on the both sides, then approximated the obtained identity with the idea of numerical integrals. Adopting the Grünwald–Letnikov formula to approximate time fractional derivative, Cui [6,7] proposed a compact difference scheme combining the compact technique in space direction. Employing the L1 approximation to discretize time derivative, Gao and Sun [9] also obtained a compact difference scheme for the one-dimensional fractional sub-diffusion equation. Zhang *et al.* proposed a Crank–Nicolson-type difference scheme for solving the sub-diffusion equation in [35], in which the H_1 norm convergence of the resultant scheme was proved and the maximum norm error estimate was obtained. For Equation (1) with nonlinear source term, Mohebbi *et al.* [23] obtained a fully discrete implicit scheme by Grünwald–Letnikov discretization of Riemann–Liouville derivative, and analysed the solvability, stability and convergence of the proposed scheme using the Fourier method. In addition, some scholars studied other numerical algorithms, such as spectral method [17,18].

An obvious fact is that, due to the non-local structure of time fractional derivatives, the computation of the solution at an instant requires information about the solution at all previous time levels, which implies a high storage requirement. One way to overcome this difficulty is to develop a high-order method. Now considerable attention has been paid to the high-order schemes. Based on the so-called block-by-block approach, Cao and Xu [3] presented a high-order scheme for the numerical solution of the fractional ordinary differential equations, and proved its convergence with the order of $3+\alpha$. Gao *et al.* [11] developed a new fractional numerical differentiation formula (called the L1-2 formula) to approximate the Caputo fractional derivative by means of the quadratic interpolation approximation. They showed that the convergence rate is $3-\alpha$. Moreover, they presented several numerical examples to demonstrate that the new L1-2 formula is much more effective and accurate than the L1 formula when solving time fractional differential equations numerically. However, they did not provide a rigorous theoretical analysis of the stability and convergence. Following the idea of L1-2 approximation, Alikhanov [1] developed the L2-1_σ scheme and presented a detailed analysis for the stability and convergence. However, the new approximation can not be applied to multi-term time fractional order equations, since the σ is dependent on α, the order of time fractional derivative. Recently, based on the idea of weighted and shifted Grünwald difference operators, Wang and Vong [31] established schemes for sub-diffusion equations and proved them by the energy method with temporal accuracy of order two and spatial accuracy of order four, respectively. Using the same idea, they also obtained a second-order scheme for diffusion-wave equations by transforming the Caputo fractional diffusion-wave equation into a Riemann–Liouville fractional integro-partial differential equation [19,27, 29].

From all papers mentioned above, we conclude that there are mainly two approaches to approximate the fractional derivatives. One is by the Grünwald definition, and the other is numerical quadrature or interpolation approximation. To our knowledge, Lubich [20,21] is first to propose the idea of fractional multi-step method to discretize the fractional calculus. And a lot of lately numerical work was motivated by his idea. Chen and Deng [5] obtained a class of fourth-order schemes by weighting and shifting Lubich's high-order operators to deal with space fractional differential equations. Zeng *et al.* developed two finite difference/element approaches for the time-fractional sub-diffusion equation with Dirichlet boundary conditions in [34], in which the time direction is approximated by the fractional linear multi-step method and the space direction is approximated by the finite element method. Very recently, Li and Ding [16] also applied the Lubich's high-order methods, second-order approximation, to solve the following reaction and anomalous-diffusion equation

$$\partial_t u(x,t) = {}_0D_t^{1-\alpha}[K_\alpha \partial_x^2 u(x,t) - C_\alpha u(x,t)] + p(x,t), \quad 0 \le t \le T, \quad 0 < x < L. \tag{4}$$

However, they only gave the analysis for $\alpha \in (\frac{3}{8}, 1)$ and claimed that their scheme is conditionally stable.

A main objective of this work is to develop a high-order method to solve two-term time fractional sub-diffusion equations. To this end, we also adopt a second-order fractional difference formula in time direction and compact technique in space direction, respectively. However, different from the Fourier method applied by Li and Ding [16], we present a convergence and stability analysis by the energy method. And it is proved that the proposed scheme is unconditionally stable and convergent with the convergence order of $O(\tau^2 + h^4)$ uniformly for $0 < \alpha, \beta < 1$, which can be seen as one of the contributions of this paper. Moreover, we extend the method to two-dimensional case and give the stability and convergence analysis as well.

The paper is organized as follows. In Section 2, we review a second-order approximation to the Riemann–Liouville time fractional derivative. In Section 3, the full discretization scheme is derived and the detailed theoretical analysis for the convergence and stability of the given schemes is provided by a prior estimate. In Section 4, we extend the compact difference scheme to the two-dimensional case and provide its convergence and stability analysis. Several numerical examples are carried out in Section 5 to verify the numerical efficiency and accuracy. Finally, we conclude the paper with some remarks in the last section.

2. A second-order difference approximation to the time fractional derivative

We begin with the definition of the Riemann–Liouville fractional derivative.

Definition 1 ([25]): *The $\gamma (0 < \gamma < 1)$ order Riemann–Liouville fractional derivative of the function $f(t)$ on $[a, T]$ is defined as*

$${}_aD_t^\gamma f(t) = \frac{1}{\Gamma(1-\gamma)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{f(s)}{(t-s)^\gamma}\,\mathrm{d}s, \quad t \in [a, T].$$

Particularly, if $a = -\infty$, we have

$${}_{-\infty}D_t^\gamma f(t) = \frac{1}{\Gamma(1-\gamma)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^t \frac{f(s)}{(t-s)^\gamma}\,\mathrm{d}s.$$

Using the fractional linear multi-step methods, Lubich obtained the L-th order ($L \le 6$) approximation of the γth derivative ($\gamma > 0$) or integral ($\gamma < 0$) by the expansion coefficients of the generating functions $\delta_L^\gamma(z)$, where

$$\delta_L^\gamma(z) = \left[\sum_{i=1}^{L} \frac{1}{i}(1-z)^i\right]^\gamma. \tag{5}$$

In [25], the first-order Grünwald difference formula,

$$A_\tau^\gamma f(t) = \frac{1}{\tau^\gamma} \sum_{k=0}^{\infty} g_k^{(\gamma)} f(t-k\tau), \tag{6}$$

was used to approximate the Riemann–Liouville fractional derivative uniformly, that is,

$$A_\tau^\gamma f(t) = {}_{-\infty}D_t^\gamma f(t) + O(\tau), \tag{7}$$

where $g_k^{(\gamma)} = (-1)^k \binom{\gamma}{k}$. In fact, the coefficients $\{g_k^{(\gamma)}\}$ in Equation (6) are the coefficients of the power series of the function $(1-z)^\gamma$, with the case $L = 1$ in Equation (5). That is,

$$(1-z)^\gamma = \sum_{k=0}^{\infty} (-1)^k \binom{\gamma}{k} z^k = \sum_{k=0}^{\infty} g_k^{(\gamma)} z^k, \tag{8}$$

for all $|z| < 1$, and they can be evaluated recursively

$$g_0^{(\gamma)} = 1, \quad g_k^{(\gamma)} = \left(1 - \frac{\gamma+1}{k}\right) g_{k-1}^{(\gamma)}, \quad k = 1, 2, \ldots. \tag{9}$$

In order to achieve a second-order approximation of the Riemann–Liouville fractional derivative, we take $L = 2$ in Equation (5) and get the following second-order Lubich approximation to Riemann–Liouville fractional derivative given by

$${}_{-\infty}D_t^\gamma f(t) \approx \tau^{-\gamma} \sum_{k=0}^{+\infty} l_k^{(\gamma)} f(t - k\tau), \tag{10}$$

where $l_k^{(\gamma)}$ are expansion coefficients of the generating function

$$\delta_2^\gamma(z) = \left[(1-z) + \frac{1}{2}(1-z)^2\right]^\gamma = (1-z)^\gamma \left[1 + \frac{1}{2}(1-z)\right]^\gamma = \sum_{k=0}^{\infty} l_k^{(\gamma)} z^k, \tag{11}$$

with

$$l_k^{(\gamma)} = (-1)^k \left(\frac{3}{2}\right)^\gamma \sum_{m=0}^{k} 3^{-m} \binom{\gamma}{k-m} \binom{\gamma}{m} = \left(\frac{3}{2}\right)^\gamma \sum_{m=0}^{k} 3^{-m} g_m^{(\gamma)} g_{k-m}^{(\gamma)}. \tag{12}$$

The above approach is suitable for a fixed value of γ to compute the coefficients. However, for various values of γ, the recurrence relationships above are not appropriate. Instead, the fast Fourier transform method [13] can be used.

Substituting $z = \mathrm{e}^{-\mathrm{i}\omega}$, we have

$$\delta_2^\gamma(\mathrm{e}^{-\mathrm{i}\omega}) = \sum_{k=0}^{\infty} l_k^{(\gamma)} \mathrm{e}^{-\mathrm{i}k\omega},$$

and coefficients $l_k^{(\gamma)}$ are expressed in terms of the Fourier transform:

$$l_k^{(\gamma)} = \frac{1}{2\pi} \int_0^{2\pi} \delta_2^\gamma(\mathrm{e}^{-\mathrm{i}\omega}) \mathrm{e}^{\mathrm{i}k\omega}\, \mathrm{d}\omega.$$

We shall be in a position to give a second-order difference formula for the Riemann–Liouville derivative with the help of the following lemma.

Lemma 2.1 ([16]): *Let* $f(t)$, ${}_{-\infty}D_t^{\gamma+2}f(t)$ *and their Fourier transforms belong to*

$$L^1(\mathbb{R}) = \left\{ f(t) \middle| \int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t < \infty \right\}$$

and denote

$$\delta_t^{\gamma} f(t) = \tau^{-\gamma} \sum_{k=0}^{+\infty} l_k^{(\gamma)} f(t - k\tau), \tag{13}$$

where $l_k^{(\gamma)}$ *is defined by Equation (12). Then*

$${}_{-\infty}D_t^{\gamma} f(t) = \delta_t^{\gamma} f(t) + O(\tau^2) \tag{14}$$

holds uniformly for $t \in \mathbb{R}$.

Remark 1: Denote

$$\mathscr{C}^{n+\alpha}(\mathbb{R}) = \left\{ f \middle| f \in L_1(\mathbb{R}), \int_{-\infty}^{+\infty} (1 + |k|)^{n+\alpha} |\hat{f}(k)|\,\mathrm{d}k < \infty \right\},$$

where $\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x$ is the Fourier transformation of $f(x)$. Consider a well defined function $f(t)$ on the bounded interval $[0, T]$. If $f^{(k)}(0) = 0$ $(k = 0, 1, 2)$, the function $f(t)$ can be extended to be defined on $\mathbb{R}$ as follows

$$\tilde{f}(t) = \begin{cases} 0, & t < 0, \\ f(t), & 0 \le t \le T, \\ v(t), & T < t < 2T, \\ 0, & t \ge 2T, \end{cases}$$

where $v(t)$ is a smooth function satisfying $v^{(k)}(T) = f^{(k)}(t)|_{t=T},\ v^{(k)}(2T) = 0,\ k = 0, 1, 2$. Suppose $\hat{f} \in \mathscr{C}^{2+\alpha}(\mathbb{R})$. Then the γ order Riemann–Liouville fractional derivatives of $f(t)$ at each point t can be approximated by

$${}_0D_t^{\gamma} f(t) = \delta_t^{\gamma} f(t) + O(\tau^2) = \tau^{-\gamma} \sum_{k=0}^{[t/\tau]} l_k^{(\gamma)} f(t - k\tau) + O(\tau^2). \tag{15}$$

It should be noted that the condition $f^{(k)}(0) = 0$ $(k = 0, 1, 2)$ may be more or less restrictive in practical applications. However, we can overcome this difficulty by making transformation of the equation itself. For more details, see [8].

3. The numerical solution to the one-dimensional fractional sub-diffusion equation

We are now ready to establish our high-order compact difference schemes.

3.1. Derivation of the compact difference scheme

In the following analysis of the numerical method, we assume the problem (1)–(3) has a unique and sufficiently smooth solution.

We partition the intervals $[0, L]$ and $[0, T]$ into a uniform mesh with the space step-size $h = L/M$ and time step-size $\tau = T/N$, respectively. Here M, N are two positive integers. And the set of grid points are denoted by $x_i = ih$ $(0 \le i \le M)$ and $t_n = n\tau$ $(0 \le n \le N)$. Denote

$$\Omega_h = \{x_i | 0 \le i \le M\}, \quad \Omega_\tau = \{t_n | 0 \le n \le N\}.$$

For any grid function $v = (v_0, v_1, \ldots, v_M)$ on Ω_h, denote

$$\delta_x v_{i-1/2} = \frac{1}{h}(v_i - v_{i-1}), \quad \delta_x^2 v_i = \frac{1}{h}(\delta_x v_{i+1/2} - \delta_x v_{i-1/2}).$$

If $u = (u^0, u^1, \ldots, u^N)$ is a grid function on Ω_τ, introduce the notations

$$u^{n+1/2} = \frac{1}{2}(u^n + u^{n+1}), \quad \delta_t u^{n+1/2} = \frac{1}{\tau}(u^{n+1} - u^n).$$

To construct a compact difference scheme for solving Equations (1)–(3), the following lemmas are needed.

Lemma 3.1 ([12]): (I) *If* $f(t) \in C^3[t_n, t_{n+1}]$, $0 \le n \le N-1$, *it holds that*

$$f'(t_{n+1/2}) = \frac{f(t_{n+1}) - f(t_n)}{\tau} - \frac{\tau^2}{16}\int_0^1 \left[f'''\left(t_{n+1/2} + \frac{\mu\tau}{2}\right) + f'''\left(t_{n+1/2} - \frac{\mu\tau}{2}\right)\right](1-\mu)^2 \, \mathrm{d}\mu,$$

where $t_{n+1/2} = (n + \frac{1}{2})\tau$.

(II) *If* $f(t) \in C^2[t_n, t_{n+1}]$, $0 \le n \le N-1$, *it holds that*

$$f(t_{n+1/2}) = \frac{f(t_{n+1}) + f(t_n)}{2} - \frac{\tau^2}{8}\int_0^1 \left[f''\left(t_{n+1/2} + \frac{\mu\tau}{2}\right) + f''\left(t_{n+1/2} - \frac{\mu\tau}{2}\right)\right](1-\mu) \, \mathrm{d}\mu.$$

Lemma 3.2 ([12]): *Denote* $\theta(s) = (1-s)^3[5 - 3(1-s)^2]$. *If* $g(x) \in C^6[x_{i-1}, x_{i+1}]$, $x_{i+1} = x_i + h$, $x_{i-1} = x_i - h$, *it holds that*

$$\frac{1}{12}[g''(x_{i-1}) + 10g''(x_i) + g''(x_{i+1})] = \frac{g(x_{i-1}) - 2g(x_i) + g(x_{i+1})}{h^2} + \frac{h^4}{360}\int_0^1 [g^{(6)}(x_i - sh) + g^{(6)}(x_i + sh)]\theta(s) \, \mathrm{d}s,$$

$$1 \le i \le M-1.$$

Introduce the average operator $\mathcal{A}$ as

$$\mathcal{A}v_i = \begin{cases} \frac{1}{12}(v_{i-1} + 10v_i + v_{i+1}), & 1 \le i \le M-1, \\ v_i, & i = 0 \text{ or } M. \end{cases}$$

It is easy to check that

$$\mathcal{A}v_i = \left(I + \frac{h^2}{12}\delta_x^2\right)v_i, \quad 1 \le i \le M-1,$$

where I denotes the unit operator.

We are now going to derive the compact difference scheme. Obviously, Equation (1) is equivalent to the following system

$$\partial_t u = (K_1 \, {}_0D_t^{\alpha} + K_2 \, {}_0D_t^{\beta})w + p, \tag{16}$$

$$w = \partial_x^2 u. \tag{17}$$

Define grid functions

$$U_i^n = u(x_i, t_n), \quad W_i^n = w(x_i, t_n), \quad p_i^n = p(x_i, t_n), \quad 0 \le i \le M, \quad 0 \le n \le N.$$

For any fixed $x \in [0, L]$, define the function

$$\tilde{u}(t) = \begin{cases} 0, & t < 0, \\ u(x,t), & 0 \le t \le T, \\ v(t), & T < t < 2T, \\ 0, & t \ge 2T, \end{cases}$$

where $v(t)$ is a smooth function satisfying $v^{(k)}(T) = \partial_t^k u(x,t)|_{t=T}$, $v^{(k)}(2T) = 0$, $k = 0, 1, 2$. Suppose $u(x,t) \in C^{(6,3)}([0,L] \times [0,T])$ and the extended function $\tilde{u}(t) \in \mathscr{C}^{2+\alpha}(\mathbb{R})$. Considering Equations (16)–(17) at the grid points (x_i, t_n), we have

$$\partial_t u(x_i, t_n) = (K_1 \, {}_0D_t^{\alpha} + K_2 \, {}_0D_t^{\beta})w(x_i, t_n) + p(x_i, t_n), \quad 0 \le i \le M, \quad 0 \le n \le N,$$

$$w(x_i, t_n) = \partial_x^2 u(x_i, t_n), \quad 0 \le i \le M, \quad 0 \le n \le N.$$

For the space discretization, performing the average operator $\mathcal{A}$ on both sides of two equalities above, yields

$$\mathcal{A}\partial_t u(x_i, t_n) = \mathcal{A}(K_1 \, {}_0D_t^{\alpha} + K_2 \, {}_0D_t^{\beta})w(x_i, t_n) + \mathcal{A}p(x_i, t_n), \quad 1 \le i \le M-1, \ 0 \le n \le N,$$

$$\mathcal{A}w(x_i, t_n) = \mathcal{A}\partial_x^2 u(x_i, t_n), \quad 1 \le i \le M-1, \ 0 \le n \le N.$$

It follows from Lemma 3.2 that

$$\mathcal{A}W_i^n = \delta_x^2 U_i^n + (R_x)_i^n, \quad 1 \le i \le M-1, \ 0 \le n \le N, \tag{18}$$

where

$$(R_x)_i^n = \frac{h^4}{360} \int_0^1 [\partial_x^6 u(x_i - sh, t_n) + \partial_x^6 u(x_i + sh, t_n)]\theta(s) \, \mathrm{d}s.$$

For the time discretization, due to Lemma 2.1, we have

$$\mathcal{A}u_t(x_i, t_n) = (K_1 \delta_t^{\alpha} + K_2 \delta_t^{\beta})\mathcal{A}W_i^n + \mathcal{A}p(x_i, t_n) + O(\tau^2), \quad 1 \le i \le M-1, 0 \le n \le N. \tag{19}$$

Averaging Equations (18)–(19) at t_n and t_{n+1}, it follows from Lemma 3.1 that

$$\mathcal{A}W_i^{n+1/2} = \delta_x^2 U_i^{n+1/2} + (R_x)_i^{n+1/2}, \quad 1 \le i \le M-1, 0 \le n \le N-1 \tag{20}$$

$$\mathcal{A}\delta_t U_i^{n+1/2} = (K_1 \delta_t^{\alpha} + K_2 \delta_t^{\beta})\mathcal{A}W_i^{n+1/2} + \mathcal{A}p_i^{n+1/2} + O(\tau^2),$$
$$1 \le i \le M-1, 0 \le n \le N-1. \tag{21}$$

Substituting Equation (20) into Equation (21) and noticing Equation (15) yield

$$\mathcal{A}\delta_t U_i^{n+1/2} = (K_1 \delta_t^{\alpha} + K_2 \delta_t^{\beta})\delta_x^2 U_i^{n+1/2} + \mathcal{A}p_i^{n+1/2} + R_i^n, \quad 1 \le i \le M-1, 0 \le n \le N-1, \tag{22}$$

where there exists a constant c_u independent of τ and h such that

$$|R_i^n| \le c_u(\tau^2 + h^4), \quad 0 \le i \le M, 0 \le n \le N-1. \tag{23}$$

Omitting the small term R_i^n in Equation (22) and denoting by u_i^n the numerical approximation of U_i^n, noticing the initial-boundary conditions (2)–(3)

$$U_0^n = \varphi_0(t_n), \quad U_M^n = \varphi_1(t_n), \quad 1 \le n \le N, \tag{24}$$

$$U_i^0 = 0, \quad 0 \le i \le M, \tag{25}$$

we get the compact difference scheme

$$\mathcal{A}\delta_t u_i^{n+1/2} = (K_1\delta_t^\alpha + K_2\delta_t^\beta)\delta_x^2 u_i^{n+1/2} + \mathcal{A}p_i^{n+1/2}, \quad 1 \le i \le M-1, 0 \le n \le N-1, \tag{26}$$

$$u_0^n = \varphi_0(t_n), \quad u_M^n = \varphi_1(t_n), \quad 1 \le n \le N, \tag{27}$$

$$u_i^0 = 0, \quad 0 \le i \le M. \tag{28}$$

Remark 2: When $\alpha = 0$, $\beta = 0$, the scheme (26)–(28) corresponds to the classical compact difference scheme for the heat equation.

3.2. Analysis of the compact difference scheme

Next we shall give the stability and convergence analysis for the scheme (26)–(28). Let

$$V_h = \{v | v \text{ is a grid function on } \Omega_h \text{ and } v_0 = v_M = 0\}.$$

For any $u, v \in V_h$, we define the discrete inner products

$$(u, v) = h\sum_{i=1}^{M-1} u_i v_i, \qquad \langle \delta_x u, \delta_x v\rangle = h\sum_{i=1}^{M} \delta_x u_{i-1/2} \cdot \delta_x v_{i-1/2},$$

and induced norms

$$\|u\| = \sqrt{(u, u)}, \qquad |u|_1 = \sqrt{\langle \delta_x u, \delta_x u\rangle}.$$

Denote maximum norm by

$$\|u\|_\infty = \max_{0\le i\le M} |u_i|.$$

It is easy to check that

$$(\delta_x^2 u, v) = -\langle \delta_x u, \delta_x v\rangle. \tag{29}$$

Some additional lemmas are still needed in order to prove the stability and convergence of the scheme (26)–(28).

Lemma 3.3: *For any $u, v \in V_h$, it holds that*

$$(\mathcal{A}u, v) = (u, \mathcal{A}v).$$

Proof: Noticing that $\mathcal{A} = I + (1/12)h^2\delta_x^2$, we have

$$(\mathcal{A}u, v) = (u, v) + \frac{1}{12}h^2(\delta_x^2 u, v) = (u, v) - \frac{1}{12}h^2\langle \delta_x u, \delta_x v\rangle$$
$$= (u, v) + \frac{1}{12}h^2(u, \delta_x^2 v) = (u, \mathcal{A}v).$$

■

Lemma 3.4: *For any $v \in V_h$, it holds that*

$$\frac{2}{3}\|v\|^2 \le (\mathcal{A}v, v) \le \|v\|^2.$$

Proof: Similar to the above lemma, we have

$$(\mathcal{A}v, v) = (v, v) + \frac{1}{12}h^2(\delta_x^2 v, v) = \|v\|^2 - \frac{1}{12}h^2|v|_1^2 \le \|v\|^2.$$

Using the inverse estimate $|v|_1^2 \le 4/h^2\|v\|^2$, we get

$$(\mathcal{A}v, v) = \|v\|^2 - \frac{1}{12}h^2|v|_1^2 \ge \|v\|^2 - \frac{4}{12}\|v\|^2 = \frac{2}{3}\|v\|^2.$$

The proof is completed.

From the above lemma, we can define the equivalent norm $\|\cdot\|$ as

$$\|v\|_A = \sqrt{(\mathcal{A}v, v)}, \tag{30}$$

and it follows that

$$\frac{2}{3}\|v\|^2 \le \|v\|_A^2 \le \|v\|^2. \tag{31}$$

■

The following lemma plays an important role in the analysis.

Lemma 3.5: *Let $\{l_k^{(\alpha)}\}_{k=0}^{\infty}$ be defined as in Equation* (12), $0 < \alpha \le 1$ *then for any positive integer* m *and real vector $(v_0, v_1, \dots, v_m)^T \in \mathbb{R}^{m+1}$, it holds that*

$$\sum_{n=0}^{m}\left(\sum_{k=0}^{n} l_k^{(\alpha)} v_{n-k}\right) v_n \ge 0.$$

Proof: To simplify the proof, we denote $l_k := l_k^{(\alpha)}$ without ambiguity. Introduce the matrix W as following

$$W = \begin{pmatrix} l_0 & \frac{l_1}{2} & \frac{l_2}{2} & \cdots & \frac{l_m}{2} \\ \frac{l_1}{2} & l_0 & \frac{l_1}{2} & \ddots & \vdots \\ \frac{l_2}{2} & \frac{l_1}{2} & \ddots & \ddots & \frac{l_2}{2} \\ \vdots & \ddots & \ddots & l_0 & \frac{l_1}{2} \\ \frac{l_m}{2} & \cdots & \frac{l_2}{2} & \frac{l_1}{2} & l_0 \end{pmatrix}.$$

To prove above quadratic form is non-negative, it is suffice to prove the symmetric Toeplitz matrix W is positive semi-definite. Notice that the generating function (see [30]) of W is given by

$$f(\alpha, x) = l_0 + \frac{1}{2}\sum_{k=1}^{\infty} l_k e^{ikx} + \frac{1}{2}\sum_{k=1}^{\infty} l_k e^{-ikx}.$$

Recalling equality (11), we have

$$f(\alpha,x)=\frac{1}{2}[\delta^{\alpha}(\mathrm{e}^{\mathrm{i}x})+\delta^{\alpha}(\mathrm{e}^{-\mathrm{i}x})]=\frac{1}{2}(1-\mathrm{e}^{\mathrm{i}x})^{\alpha}\left[1+\frac{1}{2}(1-\mathrm{e}^{\mathrm{i}x})\right]^{\alpha} + \frac{1}{2}(1-\mathrm{e}^{-\mathrm{i}x})^{\alpha}\left[1+\frac{1}{2}(1-\mathrm{e}^{-\mathrm{i}x})\right]^{\alpha}.$$

As mentioned in [4], we only need to consider the principal value of $f(\alpha,x)$ for $x\in[0,\pi]$. By calculation, we obtain

$$(1-\mathrm{e}^{\pm ix})^{\alpha}=\left(2\sin\frac{x}{2}\right)^{\alpha}\mathrm{e}^{\pm\mathrm{i}\alpha((x/2)-(\pi/2))},$$
$$\left(1+\frac{1}{2}(1-\mathrm{e}^{\pm ix})\right)^{\alpha}=\left(1+3\sin^2\frac{x}{2}\right)^{\alpha/2}\mathrm{e}^{\pm\mathrm{i}\alpha\theta},$$

where

$$\theta=\arctan\left(\frac{\sin x}{\cos x-3}\right)\in\left(-\frac{\pi}{6},0\right].$$

It follows that

$$f(\alpha,x)=\left(2\sin\frac{x}{2}\right)^{\alpha}\left(1+3\sin^2\frac{x}{2}\right)^{\alpha/2}\cos\left[\alpha\left(\frac{x}{2}-\frac{\pi}{2}+\theta\right)\right].$$

It is not difficult to check that

$$\tan\left(\frac{x}{2}\right)=\frac{\sin x}{1+\cos x}\geq\frac{\sin x}{3-\cos x}=\tan(-\theta)\geq 0,\quad x\in[0,\pi].$$

Thus $x/2\geq-\theta$ for $x\in[0,\pi]$.

Since

$$-\frac{\pi}{2}\leq\frac{x}{2}-\frac{\pi}{2}+\theta\leq\frac{\pi}{2},\quad x\in[0,\pi],$$

we have $f(\alpha,x)\geq 0$ for $\alpha\in(0,1)$. The desired result follows as a result of the Grenander–Szegö Theorem [4]. The proof is completed. ■

Now we turn to the stability and convergence analysis of schemes (26)–(28). To this end, a prior estimate will be given to simplify the proof.

Lemma 3.6: *Suppose that $\{v_i^n|0\leq i\leq M, 0\leq n\leq N\}$ is the solution of the following difference scheme*

$$\mathcal{A}\delta_t v_i^{n+1/2}-(K_1\delta_t^{\alpha}+K_2\delta_t^{\beta})\delta_x^2 v_i^{n+1/2}=S_i^n,\quad 1\leq i\leq M-1,\quad 0\leq n\leq N-1, \tag{32}$$
$$v_0^n=0,\quad v_M^n=0,\quad 1\leq n\leq N-1, \tag{33}$$
$$v_i^0=\psi_i,\quad 0\leq i\leq M, \tag{34}$$

then

$$\|v^m\|^2\leq 3\exp(3T)(\tau\sum_{n=0}^{m-1}\|S^n\|^2+\|\psi\|^2),\quad 1\leq m\leq N.$$

Proof: Making the inner product of (32) with $v^{n+1/2}$, we have

$$(\mathcal{A}\delta_t v^{n+1/2}, v^{n+1/2}) - ((K_1\delta_t^{\alpha} + K_2\delta_t^{\beta})\delta_x^2 v^{n+1/2}, v^{n+1/2}) = (S^n, v^{n+1/2}), \quad 0 \le n \le N-1. \quad (35)$$

For the first term on the left-hand side, from Lemmas 3.3 and 3.4, we obtain

$$(\mathcal{A}\delta_t v^{n+1/2}, v^{n+1/2}) = \frac{1}{2\tau}(\|v^{n+1}\|_A^2 - \|v^n\|_A^2). \quad (36)$$

For the second term on the left-hand side, noting Equation (29), we have

$$-((K_1\delta_t^{\alpha} + K_2\delta_t^{\beta})\delta_x^2 v^{n+1/2}, v^{n+1/2}) = \langle (K_1\delta_t^{\alpha} + K_2\delta_t^{\beta})\delta_x v^{n+1/2}, \delta_x v^{n+1/2}\rangle. \quad (37)$$

As to the term on the right-hand side, we get

$$(S^n, v^{n+1/2}) \le \|S^n\| \cdot \|v^{n+1/2}\|. \quad (38)$$

Substituting Equations (36)–(38) into Equation (35) and summing up the obtained inequality for n from 0 to $m-1$ lead to

$$\begin{aligned} \frac{1}{2\tau}\left(\|v^m\|_A^2 - \|v^0\|_A^2\right) &+ \sum_{n=0}^{m-1}\langle (K_1\delta_t^{\alpha} + K_2\delta_t^{\beta})\delta_x v^{n+1/2}, \delta_x v^{n+1/2}\rangle \\ &\le \sum_{n=0}^{m-1}\|S^n\| \cdot \|v^{n+1/2}\|, \quad 1 \le m \le N. \end{aligned}$$

It follows from Lemma 3.5 that

$$\begin{aligned} &\sum_{n=0}^{m-1}\langle (K_1\delta_t^{\alpha} + K_2\delta_t^{\beta})\delta_x v^{n+1/2}, \delta_x v^{n+1/2}\rangle \\ &= h\sum_{i=0}^{M-1}\left(K_1\tau^{-\alpha}\sum_{n=0}^{m-1}\sum_{k=0}^{n} l_k^{(\alpha)}\delta_x v_{i+1/2}^{n-k+\frac{1}{2}} \cdot \delta_x v_{i+1/2}^{n+1/2}\right. \\ &\quad \left. + K_2\tau^{-\beta}\sum_{n=0}^{m-1}\sum_{k=0}^{n} l_k^{(\beta)}\delta_x v_{i+1/2}^{n-k+\frac{1}{2}} \cdot \delta_x v_{i+1/2}^{n+1/2}\right) \\ &\ge 0. \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{1}{2\tau}(\|v^m\|_A^2 - \|v^0\|_A^2) &\le \sum_{n=0}^{m-1}\|S^n\| \cdot \|v^{n+1/2}\|, \\ &\le \frac{1}{2}\sum_{n=0}^{m-1}(\|S^n\|^2 + \|v^{n+1/2}\|^2), \quad 1 \le m \le N. \end{aligned}$$

Utilizing the inequality (31) , we have

$$\begin{aligned} \frac{2}{3}\|v^m\|^2 &\le \tau\sum_{n=0}^{m-1}(\|S^n\|^2 + \|v^{n+1/2}\|^2) + \|v^0\|_A^2 \\ &\le \tau\sum_{n=0}^{m-1}\|v^n\|^2 + \frac{\tau}{2}\|v^m\|^2 + \tau\sum_{n=0}^{m-1}\|S^n\|^2 + \|v^0\|^2, 1 \le m \le N. \end{aligned}$$

Consequently, when $\tau \leq \frac{2}{3}$,

$$\|v^m\|^2 \leq 3\tau \sum_{n=0}^{m-1} \|v^n\|^2 + 3\tau \sum_{n=0}^{m-1} \|S^n\|^2 + 3\|v^0\|^2, \quad 1 \leq m \leq N.$$

Then the desired result follows by the discrete Gronwall inequality. The proof is completed. ■

With the above prior estimate, the following result is easily obtained.

Theorem 3.7: *The difference scheme* (26)–(28) *is unconditionally stable to the initial value and the right-hand term for all* $0 < \alpha, \beta < 1$.

We now consider the convergence of the scheme (26)–(28).

Theorem 3.8: *Let* $\{U_i^n | 0 \leq i \leq M, 0 \leq n \leq N\}$ *be the exact solution of problem* (1)–(3), *and* $\{u_i^n\}$ *be the solution of difference schemes* (26)–(28). *Then there exists a constant c independent of* τ *and* h *such that the estimate*

$$\|U^n - u^n\| \leq c(\tau^2 + h^4), \quad 1 \leq n \leq N,$$

holds for all $0 < \alpha, \beta < 1$. *More precisely,* $c = \sqrt{3TL}\exp(3T/2)c_u$.

Proof: Let

$$e_i^n = U_i^n - u_i^n, \quad 0 \leq i \leq M, \quad 0 \leq n \leq N.$$

Subtracting Equations (26)–(28) from Equations (22), (24)–(25), we get the error system of equations:

$$\mathcal{A}\delta_t e_i^{n+1/2} - (K_1\delta_t^\alpha + K_2\delta_t^\beta)\delta_x^2 e_i^{n+1/2} = R_i^n,\ 1 \leq i \leq M-1,\ 0 \leq n \leq N-1, \tag{39}$$

$$e_0^n = 0, \quad e_M^n = 0, \quad 1 \leq n \leq N, \tag{40}$$

$$e_i^0 = 0, \quad 0 \leq i \leq M. \tag{41}$$

It follows from Lemma 3.6 and the truncation error bound (23) that

$$\|e^n\|^2 \leq \exp(3T)\left(3\tau \sum_{l=0}^{n-1} \|R^l\|^2\right)$$
$$\leq 3TL\exp(3T)c_u^2(\tau^2 + h^4)^2 = c^2(\tau^2 + h^4)^2, \quad 1 \leq n \leq N.$$

This completes the proof. ■

4. Extension to the two-dimensional fractional sub-diffusion equation

In this section, we consider the numerical algorithm and analysis for solving the following two-dimensional equation:

$$\partial_t u(x,y,t) = (K_1\, {}_0D_t^\alpha + K_2\, {}_0D_t^\beta)\Delta u(x,y,t) + p(x,y,t), (x,y) \in \Omega, 0 < t \leq T \tag{42}$$

with the initial-boundary conditions

$$u(x,y,t) = \varphi(x,y,t), \quad (x,y) \in \partial\Omega, \quad 0 < t \leq T, \tag{43}$$

$$u(x,y,0) = 0, \quad (x,y) \in \bar{\Omega}, \tag{44}$$

where $\Omega = (0, L_1) \times (0, L_2)$, $\partial\Omega$ is the boundary of Ω and Δ is the Laplace operator. In the subsequent analysis of the numerical method, we assume the problem (42)–(44) has a unique and sufficiently smooth solution.

For spatial approximation, take two integers M_1, M_2 and let $h_1 = L_1/M_1$, $h_2 = L_2/M_2$, $x_i = ih_1$ $(0 \le i \le M_1)$, $y_j = jh_2$ $(0 \le j \le M_2)$. Let $\bar{\Omega}_h = \{(x_i, y_j) | 0 \le i \le M_1, 0 \le j \le M_2\}$, and $\Omega_h = \bar{\Omega}_h \cap \Omega$, and $\partial\Omega_h = \bar{\Omega}_h \cap \partial\Omega$. For any grid function $v = \{v_{i,j} | 0 \le i \le M_1, 0 \le j \le M_2\}$, denote

$$\delta_x v_{i-1/2,j} = \frac{1}{h_1}(v_{i,j} - v_{i-1,j}), \quad \delta_x^2 v_{i,j} = \frac{1}{h_1}(\delta_x v_{i+1/2,j} - \delta_x v_{i-1/2,j}).$$

Similar notations $\delta_y v_{i,j-(1/2)}$, $\delta_y^2 v_{i,j}$ can be defined. The spatial average operators are defined as

$$\mathcal{A}_1 v_{i,j} = \begin{cases} \frac{1}{12}(v_{i-1,j} + 10v_{i,j} + v_{i+1,j}), & 1 \le i \le M_1 - 1, \\ v_{i,j}, & i = 0 \text{ or } M_1, \end{cases} \quad 0 \le j \le M_2.$$

$$\mathcal{A}_2 v_{i,j} = \begin{cases} \frac{1}{12}(v_{i,j-1} + 10v_{i,j} + v_{i,j+1}), & 1 \le j \le M_2 - 1, \\ v_{i,j}, & j = 0 \text{ or } M_2, \end{cases} \quad 0 \le i \le M_1.$$

In addition, denote $\Delta_h u_{i,j} = \delta_x^2 u_{i,j} + \delta_y^2 u_{i,j}$.

Let

$$w = \Delta u,$$

then Equation (42) is equivalent to the following system

$$\partial_t u = (K_1 \, {}_0D_t^{\alpha} + K_2 \, {}_0D_t^{\beta}) w + p, \tag{45}$$

$$w = \Delta u. \tag{46}$$

Denote

$$U_{i,j}^n = u(x_i, y_j, t_n), \quad W_{i,j}^n = w(x_i, y_j, t_n), \quad p_{i,j}^n = p(x_i, y_j, t_n), \ (x_i, y_j) \in \bar{\Omega}_h, \quad 0 \le n \le N.$$

Suppose $u(x, y, t) \in C^{(6,6,3)}([0, L_1] \times [0, L_2] \times [0, T])$ and the extended function, similarly defined as in Section 3, $\tilde{u}(t) \in \mathscr{C}^{2+\alpha}(\mathbb{R})$. Considering Equations (45)–(46) at the grid points (x_i, y_j, t_n), we have

$$\partial_t u(x_i, y_j, t_n) = (K_1 \, {}_0D_t^{\alpha} + K_2 \, {}_0D_t^{\beta}) w(x_i, y_j, t_n) + p(x_i, y_j, t_n), \quad (x_i, y_j) \in \bar{\Omega}_h, \quad 0 \le n \le N,$$

$$w(x_i, y_j, t_n) = \partial_x^2 u(x_i, y_j, t_n) + \partial_y^2 u(x_i, y_j, t_n), \quad (x_i, y_j) \in \bar{\Omega}_h, \quad 0 \le n \le N.$$

For the space discretization, performing the average operators $\mathcal{A}_1$ and $\mathcal{A}_2$ on both sides of the above equalities yields

$$\mathcal{A}_1\mathcal{A}_2 \partial_t u(x_i, y_j, t_n) = \mathcal{A}_1\mathcal{A}_2 (K_1 \, {}_0D_t^{\alpha} + K_2 \, {}_0D_t^{\beta}) w(x_i, y_j, t_n) + \mathcal{A}_1\mathcal{A}_2 p(x_i, y_j, t_n), \quad (x_i, y_j) \in \Omega_h, \quad 0 \le n \le N,$$

$$\mathcal{A}_1\mathcal{A}_2 w(x_i, y_j, t_n) = \mathcal{A}_2[\mathcal{A}_1 \partial_x^2 u(x_i, y_j, t_n)] + \mathcal{A}_1\big[\mathcal{A}_2 \partial_y^2 u(x_i, y_j, t_n)\big], \quad (x_i, y_j) \in \Omega_h, \quad 0 \le n \le N.$$

It follows from Lemma 3.2 that

$$\mathcal{A}_1\mathcal{A}_2 W_{i,j}^n = \mathcal{A}_2 \delta_x^2 U_{i,j}^n + \mathcal{A}_1 \delta_y^2 U_{i,j}^n + \mathcal{A}_2 (R_x)_{i,j}^n + \mathcal{A}_1 (R_y)_{i,j}^n, \quad (x_i, y_j) \in \Omega_h, 0 \le n \le N. \tag{47}$$

where

$$(R_x)_{i,j}^n = \frac{h_1^4}{360}\int_0^1 [\partial_x^6 u(x_i - sh_1, y_j, t_n) + \partial_x^6 u(x_i + sh_1, y_j, t_n)]\theta(s)\, ds,$$

$$R_y)_{i,j}^n = \frac{h_2^4}{360}\int_0^1 [\partial_y^6 u(x_i, y_j - sh_2, t_n) + \partial_y^6 u(x_i, y_j + sh_2, t_n)]\theta(s)\, ds.$$

For the time discretization, due to Lemma 2.1, we have

$$\mathcal{A}_1\mathcal{A}_2 u_t(x_i, y_j, t_n) = (K_1\delta_t^\alpha + K_2\delta_t^\beta)\mathcal{A}_1\mathcal{A}_2 W_{i,j}^n + \mathcal{A}_1\mathcal{A}_2 p(x_i, y_j, t_n) + O(\tau^2),$$
$$(x_i, y_j) \in \Omega_h, \quad 0 \le n \le N. \tag{48}$$

Averaging Equations (47)–(48) at t_n and t_{n+1}, it follows from Lemma 3.1 that

$$\mathcal{A}_1\mathcal{A}_2 W_{i,j}^{n+1/2} = \mathcal{A}_2\delta_x^2 U_{i,j}^{n+1/2} + \mathcal{A}_1\delta_y^2 U_{i,j}^{n+1/2} + \mathcal{A}_2(R_x)_{i,j}^{n+1/2} + \mathcal{A}_1(R_y)_{i,j}^{n+1/2},$$
$$(x_i, y_j) \in \Omega_h, \quad 0 \le n \le N-1. \tag{49}$$

$$\mathcal{A}_1\mathcal{A}_2\delta_t U_{i,j}^{n+1/2} = (K_1\delta_t^\alpha + K_2\delta_t^\beta)\mathcal{A}_1\mathcal{A}_2 W_{i,j}^{n+1/2} + \mathcal{A}_1\mathcal{A}_2 p_{i,j}^{n+1/2} + O(\tau^2),$$
$$(x_i, y_j) \in \Omega_h, \quad 0 \le n \le N-1. \tag{50}$$

Substituting Equation (49) into Equation (50) and noticing Equation (15) yield

$$\mathcal{A}_1\mathcal{A}_2\delta_t U_{i,j}^{n+1/2} = (K_1\delta_t^\alpha + K_2\delta_t^\beta)(\mathcal{A}_2\delta_x^2 U_{i,j}^{n+1/2} + \mathcal{A}_1\delta_y^2 U_{i,j}^{n+1/2}) + \mathcal{A}_1\mathcal{A}_2 p_{i,j}^{n+1/2}$$
$$+ O(\tau^2 + h_1^4 + h_2^4),$$
$$(x_i, y_j) \in \Omega_h, 0 \le n \le N-1. \tag{51}$$

Omitting the small term and denoting by $u_{i,j}^n$ the numerical approximation of $U_{i,j}^n$, noticing the initial-boundary conditions,

$$U_{i,j}^n = \varphi(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \le n \le N, \tag{52}$$

$$U_{i,j}^0 = 0, \quad (x_i, y_j) \in \bar{\Omega}_h, \tag{53}$$

we get the compact difference scheme

$$\mathcal{A}_1\mathcal{A}_2\delta_t u_{i,j}^{n+1/2} = (K_1\delta_t^\alpha + K_2\delta_t^\beta)(\mathcal{A}_2\delta_x^2 u_{i,j}^{n+1/2} + \mathcal{A}_1\delta_y^2 u_{i,j}^{n+1/2}) + \mathcal{A}_1\mathcal{A}_2 p_{i,j}^{n+1/2},$$
$$(x_i, y_j) \in \Omega_h, \quad 0 \le n \le N-1, \tag{54}$$

$$u_{i,j}^n = \varphi(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \le n \le N, \tag{55}$$

$$u_{i,j}^0 = 0, \quad (x_i, y_j) \in \bar{\Omega}_h, \tag{56}$$

4.1. Analysis of the compact difference scheme

In order to analyse the stability and convergence of the compact difference scheme (54)–(56), similar to the previous section, we introduce discrete inner products and corresponding norms. Let

$$V_h = \{v | v \text{ is a grid function in } \bar{\Omega}_h \quad \text{and} \quad v_{i,j} = 0 \text{ if } (x_i, y_j) \in \partial\Omega_h\}.$$

For any $u, v \in V_h$, we define

$$(u, v) = h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} u_{i,j} v_{i,j},$$

$$\langle \delta_x u, \delta_x v \rangle = h_1 h_2 \sum_{i=1}^{M_1} \sum_{j=1}^{M_2-1} \delta_x u_{i-1/2,j} \cdot \delta_x v_{i-1/2,j},$$

And similarly for $\langle \delta_y u, \delta_y v \rangle$. In addition, norms are given by

$$\|u\| = \sqrt{(u, u)}, \quad \|\delta_x u\| = \sqrt{\langle \delta_x u, \delta_x u \rangle},$$
$$\|u\|_\infty = \max_{0 \le i \le M_1, 0 \le j \le M_2} |u_{i,j}|.$$

Also, $\|\delta_y u\|$ can be defined similarly. Obviously, we have

$$(\delta_x^2 u, v) = -\langle \delta_x u, \delta_x v \rangle, \quad (\delta_y^2 u, v) = -\langle \delta_y u, \delta_y v \rangle. \tag{57}$$

Lemma 4.1 ((see [10])): *For any $v \in V_h$, it holds that*

$$\tfrac{1}{3}\|v\|^2 \le (\mathcal{A}_1 \mathcal{A}_2 v, v) \le \|v\|^2.$$

Thus, we can define the equivalent norm of $\|v\|$ as

$$\|v\|_A = \sqrt{(\mathcal{A}_1 \mathcal{A}_2 v, v)},$$

that is,

$$\tfrac{1}{3}\|v\|^2 \le \|v\|_A^2 \le \|v\|^2. \tag{58}$$

A prior estimate is given to show the stability and convergence firstly.

Lemma 4.2: *Suppose $\{v_{i,j}^n | 0 \le i \le M_1,\ 0 \le j \le M_2, 0 \le n \le N\}$ is the solution of the following difference scheme*

$$\mathcal{A}_1 \mathcal{A}_2 \delta_t v_{i,j}^{n+1/2} - (K_1 \delta_t^\alpha + K_2 \delta_t^\beta)(\mathcal{A}_2 \delta_x^2 v_{i,j}^{n+1/2} + \mathcal{A}_1 \delta_y^2 v_{i,j}^{n+1/2}) = S_{i,j}^n,$$
$$(x_i, y_j) \in \Omega_h, 0 \le n \le N-1, \tag{59}$$
$$v_{i,j}^n = \varphi(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial\Omega_h, 1 \le n \le N, \tag{60}$$
$$v_{i,j}^0 = \psi_{i,j}, \quad (x_i, y_j) \in \bar{\Omega}_h, \tag{61}$$

then

$$\|v^n\|^2 \le 6 \exp(6\,T)(\tau \sum_{l=0}^{n-1} \|S^l\|^2 + \|\psi\|^2), \quad 1 \le n \le N.$$

Proof: Making the inner product of (59) with $v^{n+1/2}$, we obtain

$$(\mathcal{A}_1\mathcal{A}_2\delta_t v^{n+1/2}, v^{n+1/2}) - ((K_1\delta_t^\alpha + K_2\delta_t^\beta)(\mathcal{A}_2\delta_x^2 + \mathcal{A}_1\delta_y^2)v^{n+1/2}, v^{n+1/2})$$
$$= (S^n, v^{n+1/2}), \quad 0 \le n \le N-1. \tag{62}$$

For the first term on the left-hand side of Equation (62), from Lemma 4.1, we have

$$(\mathcal{A}_1\mathcal{A}_2\delta_t v^{n+1/2}, v^{n+1/2}) = \frac{1}{2\tau}(\|v^{n+1}\|_A^2 - \|v^n\|_A^2). \tag{63}$$

Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are positive definite and self-adjoint, we can consider their square roots denoted by Q_x and Q_y, respectively. For the second term on the left-hand side of Equation (62), observing the commutativity of operators in different spatial directions, we have

$$\begin{aligned}
&-((K_1\delta_t^\alpha + K_2\delta_t^\beta)(\mathcal{A}_2\delta_x^2 + \mathcal{A}_1\delta_y^2)v^{n+1/2}, v^{n+1/2})\\
&= -((K_1\delta_t^\alpha + K_2\delta_t^\beta)Q_y\delta_x^2 v^{n+1/2}, Q_y v^{n+1/2}) - ((K_1\delta_t^\alpha + K_2\delta_t^\beta)Q_x\delta_y^2 v^{n+1/2}, Q_x v^{n+1/2})\\
&= \langle (K_1\delta_t^\alpha + K_2\delta_t^\beta)Q_y\delta_x v^{n+1/2}, Q_y\delta_x v^{n+1/2}\rangle + \langle (K_1\delta_t^\alpha + K_2\delta_t^\beta)Q_x\delta_y v^{n+1/2}, Q_x\delta_y v^{n+1/2}\rangle.
\end{aligned} \tag{64}$$

For the term on the right-hand side of Equation (62), we get

$$(S^n, v^{n+1/2}) \le \|S^n\| \cdot \|v^{n+1/2}\|. \tag{65}$$

Substituting Equations (63)–(65) into Equation (62) yields

$$\begin{aligned}
&\frac{1}{2\tau}(\|v^{n+1}\|_A^2 - \|v^n\|_A^2) + \langle (K_1\delta_t^\alpha + K_2\delta_t^\beta)Q_y\delta_x v^{n+1/2}, Q_y\delta_x v^{n+1/2}\rangle\\
&\quad + \langle (K_1\delta_t^\alpha + K_2\delta_t^\beta)Q_x\delta_y v^{n+1/2}, Q_x\delta_y v^{n+1/2}\rangle \le \|S^n\| \cdot \|v^{n+1/2}\|.
\end{aligned}$$

Summing up for n from 0 to $m-1$ gives

$$\begin{aligned}
&\frac{1}{2\tau}(\|v^m\|_A^2 - \|v^0\|_A^2) + \sum_{n=0}^{m-1}\langle (K_1\delta_t^\alpha + K_2\delta_t^\beta)Q_y\delta_x v^{n+1/2}, Q_y\delta_x v^{n+1/2}\rangle\\
&\quad + \sum_{n=0}^{m-1}\langle (K_1\delta_t^\alpha + K_2\delta_t^\beta)Q_x\delta_y v^{n+1/2}, Q_x\delta_y v^{n+1/2}\rangle \le \sum_{n=0}^{m-1}\|S^n\| \cdot \|v^{n+1/2}\|.
\end{aligned}$$

Due to Lemma 3.5, we know that both the second term and the third term on the left-hand side of the above inequality are non-negative. It follows that

$$\begin{aligned}
\frac{1}{2\tau}(\|v^m\|_A^2 - \|v^0\|_A^2) &\le \sum_{n=0}^{m-1}\|S^n\| \cdot \|v^{n+1/2}\|\\
&\le \frac{1}{2}\sum_{n=0}^{m-1}(\|S^n\|^2 + \|v^{n+1/2}\|^2).
\end{aligned}$$

Following the similar argument as that in the previous section, we can get the desired result. This completes the proof. ∎

From the above prior estimate, it is straightforward to obtain the following result.

Theorem 4.3: *The difference scheme* (54)–(56) *is unconditionally stable to the initial value and the right-hand term for all* $0 < \alpha, \beta < 1$.

We now focus on the convergence of the difference scheme (54)–(56).

Theorem 4.4: *Let* $\{U_{i,j}^n | 0 \le i \le M_1,\ 0 \le j \le M_2,\ 0 \le n \le N\}$ *be the exact solution of problem* (42)–(44), *and* $\{u_{i,j}^n | 0 \le i \le M_1,\ 0 \le j \le M_2,\ 0 \le n \le N\}$ *be the solution of difference scheme* (54)–(56). *Then there exists a constant* c *such that the following estimate*

$$\|U^n - u^n\| \le c(\tau^2 + h_1^4 + h_2^4), \quad 1 \le n \le N,$$

holds for all $0 < \alpha, \beta < 1$.

Proof: Let

$$e_{i,j}^n = U_{i,j}^n - u_{i,j}^n, \quad 0 \le i \le M_1, 0 \le j \le M_2, 0 \le n \le N.$$

Subtracting Equations (54)–(56) from Equations (51)–(52), we get the error system of the equations:

$$\begin{aligned}
&\mathcal{A}_1\mathcal{A}_2\delta_t e_{i,j}^{n+1/2} - (K_1\delta_t^\alpha + K_2\delta_t^\beta)(\mathcal{A}_2\delta_x^2 e_{i,j}^{n+1/2} + \mathcal{A}_1\delta_y^2 e_{i,j}^{n+1/2}) = R_{i,j}^{n+1/2},\\
&\qquad (x_i, y_j) \in \Omega_h, \quad 0 \le n \le N-1,\\
&e_{i,j}^n = 0, \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \le n \le N,\\
&e_{i,j}^0 = 0, \quad (x_i, y_j) \in \bar{\Omega}_h,
\end{aligned}$$

where there exists a constant $\tilde{c}_u$ such that

$$|R_{i,j}^{n+1/2}| \le \tilde{c}_u(\tau^2 + h_1^4 + h_2^4), \quad (x_i, y_j) \in \Omega_h, \quad 0 \le n \le N-1.$$

It follows from Lemma 4.2 that

$$\begin{aligned}
\|e^n\|^2 &\le 6\exp(6\,T)\left(\tau\sum_{l=0}^{n-1}\|R^n\|^2\right)\\
&\le 6L_1L_2T\exp(6T)\tilde{c}_u^2(\tau^2 + h_1^4 + h_2^4)^2\\
&= c^2(\tau^2 + h_1^4 + h_2^4)^2, \quad 1 \le n \le N.
\end{aligned}$$

This completes the proof. ■

5. Numerical examples

In this section, we carry out numerical experiments to show the effectiveness and convergence orders of the proposed schemes.

In the runs, we compute the maximum norm errors of the numerical solution.

Let

$$E(h, \tau) = \max_{0 \le k \le N} \|U^k - u^k\|_\infty$$

and assume

$$E(h, \tau) = O(\tau^p + h^q).$$

If h is small enough, then $E(h, \tau) \approx O(\tau^p)$. Consequently, $E(h, 2\tau)/E(h, \tau) \approx 2^p$ and hence $p \approx \log_2(E(h, 2\tau)/E(h, \tau))$ is the convergence order with respect to the temporal step-size. If

Table 1. Numerical convergence order of the scheme (26)–(28) in the temporal direction with a fixed space step-size $h = \frac{1}{100}$ (Example 5.1).

α	N	$E(h,\tau)$	p
0	10	2.4999e−003	–
	20	6.2494e−004	2.0001
	40	1.5619e−004	2.0004
	80	3.9001e−005	2.0017
0.1	10	1.8255e−003	–
	20	4.5697e−004	1.9982
	40	1.1420e−004	2.0006
	80	2.8504e−005	2.0023
0.5	10	1.2033e−003	–
	20	2.1881e−004	2.4593
	40	5.2748e−005	2.0525
	80	1.3137e−005	2.0054
0.9	10	4.1536e−003	–
	20	1.0110e−003	2.0386
	40	2.4522e−004	2.0437
	80	6.0031e−005	2.0303

τ is small enough, then $E(h,\tau) \approx O(h^q)$. Consequently, $E(2h,\tau)/E(h,\tau) \approx 2^q$ and hence $q \approx \log_2(E(2h,\tau)/E(h,\tau))$ is the convergence order with respect to the spatial step-size. Denote

$$p(h,\tau) = \log_2\left(\frac{E(h,2\tau)}{E(h,\tau)}\right), \quad q(h,\tau) = \log_2\left(\frac{E(2h,\tau)}{E(h,\tau)}\right).$$

Firstly, we examine a special case of the anomalous diffusion equation with the single fractional order term.

Example 5.1: Consider the following example [31],

$$\partial_t u(x,t) = {}_0D_t^{\alpha}\partial_x^2 u(x,t) + \left[2t + \frac{8\pi^2 t^{2-\alpha}}{\Gamma(3-\alpha)}\right]\sin(2\pi x), \quad 0 < x < 1, 0 < t \le 1, \tag{66}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad 0 < t \le 1, \tag{67}$$

$$u(x,0) = 0, \quad 0 \le x \le 1, \tag{68}$$

with the exact solution given by $u(x,t) = t^2\sin(2\pi x)$.

We test the temporal errors and convergence orders by choosing different α, letting τ vary and fixing the space step-size h sufficiently small to avoid contamination of the spatial errors. Table 1 shows the numerical results when $h = 1/100$, $\tau = 1/10, 1/20, 1/40$, and $1/80$, respectively. It can be observed from Table 1 that the convergence order of the compact difference scheme (26)–(28) is about 2 with respect to the temporal step-size.

Next, we test the spatial errors and convergence orders by letting h vary and fixing the time step-size τ sufficiently small to avoid contamination of the temporal errors. Table 2 shows the numerical results when $\tau = \frac{1}{1000}$, $h = \frac{1}{4}, h = \frac{1}{8}, h = \frac{1}{16}$ and $h = \frac{1}{32}$. It can be seen from Table 2 that the convergence order of the scheme (26)–(28) is about 4 with respect to the spatial step-size.

Secondly, we consider a more general case.

Table 2. Numerical convergence order of the scheme (26)–(28) in the spatial direction with a fixed temporal step-size $\tau = \frac{1}{1000}$ (Example 5.1).

α	M	$E(h,\tau)$	q
0	4	2.6659e−002	–
	8	1.5445e−003	4.1094
	16	9.4526e−005	4.0303
	32	5.6463e−006	4.0653
0.1	4	2.6727e−002	–
	8	1.5484e−003	4.1095
	16	9.4828e−005	4.0293
	32	5.7281e−006	4.0492
0.5	4	2.7021e−002	–
	8	1.5652e−003	4.1097
	16	9.6106e−005	4.0256
	32	6.0565e−006	3.9881
0.9	4	2.7306e−002	–
	8	1.5816e−003	4.1098
	16	9.7383e−005	4.0215
	32	6.4089e−006	3.9255

Table 3. Numerical convergence order of the scheme (26)–(28) in the temporal direction with a fixed space step-size $h = 1/20$ (Example 5.2).

α	β	N	$E(h,\tau)$	p
0.05	0.35	5	2.0977e−003	–
		10	4.8697e−004	2.1069
		20	1.1660e−004	2.0623
		40	2.8483e−005	2.0333
		80	7.0362e−006	2.0172
0.3	0.7	5	6.6713e−004	–
		10	2.0076e−004	1.7325
		20	5.0812e−005	1.9823
		40	1.2920e−005	1.9755
		80	3.2693e−006	1.9826

Example 5.2:

$$\begin{aligned} &\partial_t u(x,t) = ({}_0D_t^{\alpha} + {}_0D_t^{\beta})\partial_x^2 u(x,t) + p(x,t) \quad 0 < x < 1,\ 0 < t \le 1, \\ &u(0,t) = 0, \quad u(1,t) = t^{3-\alpha-\beta}\sin 1, \quad 0 < t \le 1, \\ &u(x,0) = 0, \quad 0 \le x \le 1, \end{aligned}$$

with the source term

$$p(x,t) = [(3-\alpha-\beta)t^{2-\alpha-\beta} + \frac{\Gamma(4-\alpha-\beta)}{\Gamma(4-2\alpha-\beta)}t^{3-2\alpha-\beta} + \frac{\Gamma(4-\alpha-\beta)}{\Gamma(4-\alpha-2\beta)}t^{3-\alpha-2\beta}]\sin x.$$

It is easy to check that the exact solution is $u(x,t) = t^{3-\alpha-\beta}\sin x$.

Take $(\alpha,\beta) = (0.05, 0.35), (0.3, 0.7)$, respectively. From Table 3, we can see that the convergence order with respect to temporal step-size is about 2, which is in accordance with our theoretical results.

Finally, a two-dimensional case will be given to illustrate the correctness of the theoretical results.

Table 4. Numerical convergence order of the scheme (54)–(56) in the temporal direction with fixed space step-sizes $h_1 = h_2 = \pi/20$ (Example 5.3).

α	β	N	$E(h,\tau)$	p
0.05	0.3	5	1.4128e−002	–
		10	3.2287e−003	2.1295
		20	7.6794e−004	2.0719
		40	1.8592e−004	2.0463
		80	4.4556e−005	2.0610
0.85	0.95	5	2.5559e−002	–
		10	7.7275e−003	1.7257
		20	2.0762e−003	1.8960
		40	5.3913e−004	1.9452
		80	1.3871e−004	1.9586

Example 5.3: We consider the problem (42)–(44) with an exact solution:

$$u(x, y, t) = t^3 \sin x \sin y \tag{69}$$

in the domain $\Omega \times [0, T]$, where $K_1 = 1, K_2 = 2, \quad \Omega = (0, \pi) \times (0, \pi)$. It can be checked that the corresponding forcing term and initial-boundary conditions are, respectively,

$$\begin{aligned} p(x, y, t) &= [3t^2 + \frac{12}{\Gamma(4-\alpha)} t^{3-\alpha} + \frac{24}{\Gamma(4-\beta)} t^{3-\beta}] \sin x \sin y, \\ \varphi(x, y, t) &= 0, \quad u(x, y, 0) = 0. \end{aligned} \tag{70}$$

Table 4 shows that the convergence order with respect to temporal step-size is about 2, which is in a good agreement with our theoretical prediction.

6. Conclusion

In this paper, based on the Lubich's second-order operator in time direction and combining the compact technique in space discretization, a compact difference scheme is derived to solve the time fractional sub-diffusion equations for both one- and two-dimensional cases. The stability and convergence are proved by the energy method and convergence orders are two in temporal direction and four in spatial direction, respectively. Furthermore, several numerical examples are employed to validate our theoretical results on the proposed compact differences scheme. Our method can be easily employed to prove that the proposed scheme in [16] for Equation (4) is unconditionally stable for $0 < \alpha < 1$. In the future work, we plan to apply the proposed method to other more realistic problem.

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References

[1] A.A. Alikhanov, *A new difference scheme for the time fractional diffusion equation*, J. Comput. Phys. 280 (2015), pp. 424–438. Available at http://dx.doi.org/10.1016/j.jcp.2014.09.031.

[2] G.A. Anastassiou, *Advances on Fractional Inequalities*. Springer Briefs in Mathematics, Springer, New York, 2011. Available at http://dx.doi.org/10.1007/978-1-4614-0703-4.

[3] J. Cao and C. Xu, *A high order schema for the numerical solution of ordinary fractional differential equations*, in *Recent Advances in Scientific Computing and Applications*, Contemp. Math., Vol. 586, Amer. Math. Soc., Providence, RI, 2013, pp. 93–103. Available at http://dx.doi.org/10.1090/conm/586/11658.

[4] R.H.F. Chan and X.Q. Jin, *An Introduction to Iterative Toeplitz Solvers*. Fundamentals of Algorithms, Vol. 5, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007. Available at http://dx.doi.org/10.1137/1.9780898718850.

[5] M. Chen and W. Deng, *Fourth order accurate scheme for the space fractional diffusion equations*, SIAM J. Numer. Anal. 52 (2014), pp. 1418–1438. Available at http://dx.doi.org/10.1137/130933447.

[6] M. Cui, *Compact finite difference method for the fractional diffusion equation*, J. Comput. Phys. 228 (2009), pp. 7792–7804. Available at http://dx.doi.org/10.1016/j.jcp.2009.07.021.

[7] M. Cui, *Compact alternating direction implicit method for two-dimensional time fractional diffusion equation*, J. Comput. Phys. 231 (2012), pp. 2621–2633. Available at http://dx.doi.org/10.1016/j.jcp.2011.12.010.

[8] Y. Dimitrov, *Numerical approximations for fractional differential equations*, J. Fract. Calc. Appl. 5 (2014), pp. 22–45.

[9] G.-H. Gao and Z.-Z. Sun, *A compact finite difference scheme for the fractional sub-diffusion equations*, J. Comput. Phys. 230 (2011), pp. 586–595. Available at http://dx.doi.org/10.1016/j.jcp.2010.10.007.

[10] G.-H. Gao, H.-W. Sun, and Z.-Z. Sun, *Stability and convergence of finite difference schemes for a class of time-fractional sub-diffusion equations based on certain superconvergence*, J. Comput. Phys. 280 (2015), pp. 510–528. Available at http://dx.doi.org/10.1016/j.jcp.2014.09.033.

[11] G.-H. Gao, Z.-Z. Sun, and H.-W. Zhang, *A new fractional numerical differentiation formula to approximate the Caputo fractional derivative and its applications*, J. Comput. Phys. 259 (2014), pp. 33–50. Available at http://dx.doi.org/10.1016/j.jcp.2013.11.017.

[12] Z.-P. Hao, Z.-Z. Sun, and W.-R. Cao, *A three-level linearized compact difference scheme for the Ginzburg–Landau equation*, Numer. Methods Partial Differential Equations 31 (2015), pp. 876–899. Available at http://onlinelibrary.wiley.com/doi/10.1002/num.21925/full.

[13] P. Henrici, *Fast Fourier methods in computational complex analysis*, SIAM Rev. 21 (1979), pp. 481–527. Available at http://dx.doi.org/10.1137/1021093.

[14] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies Vol. 204, Elsevier, Amsterdam, 2006.

[15] T.A.M. Langlands and B.I. Henry, *The accuracy and stability of an implicit solution method for the fractional diffusion equation*, J. Comput. Phys. 205 (2005), pp. 719–736. Available at http://dx.doi.org/10.1016/j.jcp.2004.11.025.

[16] C. Li and H. Ding, *Higher order finite difference method for the reaction and anomalous-diffusion equation*, Appl. Math. Model. 38 (2014), pp. 3802–3821. Available at http://dx.doi.org/10.1016/j.apm.2013.12.002.

[17] X. Li and C. Xu, *A space-time spectral method for the time fractional diffusion equation*, SIAM J. Numer. Anal. 47 (2009), pp. 2108–2131. Available at http://dx.doi.org/10.1137/080718942.

[18] Y. Lin and C. Xu, *Finite difference/spectral approximations for the time-fractional diffusion equation*, J. Comput. Phys. 225 (2007), pp. 1533–1552. Available at http://dx.doi.org/10.1016/j.jcp.2007.02.001.

[19] J.C. López Marcos, *A difference scheme for a nonlinear partial integrodifferential equation*, SIAM J. Numer. Anal. 27 (1990), pp. 20–31. Available at http://dx.doi.org/10.1137/0727002.

[20] C. Lubich, *On the stability of linear multistep methods for Volterra convolution equations*, IMA J. Numer. Anal. 3 (1983), pp. 439–465. Available at http://dx.doi.org/10.1093/imanum/3.4.439.

[21] C. Lubich, *Discretized fractional calculus*, SIAM J. Math. Anal. 17 (1986), pp. 704–719. Available at http://dx.doi.org/10.1137/0517050.

[22] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, 2010. Available at http://dx.doi.org/10.1142/9781848163300, an introduction to mathematical models.

[23] A. Mohebbi, M. Abbaszadeh, and M. Dehghan, *A high-order and unconditionally stable scheme for the modified anomalous fractional sub-diffusion equation with a nonlinear source term*, J. Comput. Phys. 240 (2013), pp. 36–48. Available at http://dx.doi.org/10.1016/j.jcp.2012.11.052.

[24] K.B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York–London, 1974, theory and applications of differentiation and integration to arbitrary order, with an annotated chronological bibliography by Bertram Ross, Mathematics in Science and Engineering, Vol. 111.

[25] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, Vol. 198, Academic Press, Inc., San Diego, CA, 1999, an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.

[26] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science Publishers, Yverdon, 1993, theory and applications, edited and with a foreword by S.M. Nikol'skiĭ, translated from the 1987 Russian original, revised by the authors.
[27] J.M. Sanz-Serna, *A numerical method for a partial integro-differential equation*, SIAM J. Numer. Anal. 25 (1988), pp. 319–327. Available at http://dx.doi.org/10.1137/0725022.
[28] Z.-Z. Sun and X. Wu, *A fully discrete difference scheme for a diffusion-wave system*, Appl. Numer. Math. 56 (2006), pp. 193–209. Available at http://dx.doi.org/10.1016/j.apnum.2005.03.003.
[29] T. Tang, *A finite difference scheme for partial integro-differential equations with a weakly singular kernel*, Appl. Numer. Math. 11 (1993), pp. 309–319. Available at http://dx.doi.org/10.1016/0168-9274(93)90012-G.
[30] W. Tian, H. Zhou, and W. Deng, *A class of second order difference approximations for solving space fractional diffusion equations*, Math. Comput. 84 (2015), pp. 1703–1727.
[31] Z. Wang and S. Vong, *Compact difference schemes for the modified anomalous fractional sub-diffusion equation and the fractional diffusion-wave equation*, J. Comput. Phys. 277 (2014), pp. 1–15. Available at http://dx.doi.org/10.1016/j.jcp.2014.08.012.
[32] S.B. Yuste, *Weighted average finite difference methods for fractional diffusion equations*, J. Comput. Phys. 216 (2006), pp. 264–274. Available at http://dx.doi.org/10.1016/j.jcp.2005.12.006.
[33] S.B. Yuste and L. Acedo, *An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equations*, SIAM J. Numer. Anal. 42 (2005), pp. 1862–1874. Available at http://dx.doi.org/10.1137/030602666.
[34] F. Zeng, C. Li, F. Liu, and I. Turner, *The use of finite difference/element approaches for solving the time-fractional subdiffusion equation*, SIAM J. Sci. Comput. 35 (2013), pp. A2976–A3000. Available at http://dx.doi.org/10.1137/130910865.
[35] Y.-N. Zhang, Z.-Z. Sun, and H.-W. Wu, *Error estimates of Crank–Nicolson-type difference schemes for the subdiffusion equation*, SIAM J. Numer. Anal. 49 (2011), pp. 2302–2322. Available at http://dx.doi.org/10.1137/100812707.
[36] P. Zhuang, F. Liu, V. Anh, and I. Turner, *New solution and analytical techniques of the implicit numerical method for the anomalous subdiffusion equation*, SIAM J. Numer. Anal. 46 (2008), pp. 1079–1095. Available at http://dx.doi.org/10.1137/060673114.
[37] P. Zhuang, F. Liu, V. Anh, and I. Turner, *Stability and convergence of an implicit numerical method for the non-linear fractional reaction–subdiffusion process*, IMA J. Appl. Math. 74 (2009), pp. 645–667. Available at http://dx.doi.org/10.1093/imamat/hxp015.