Study Guide for Exam 3

1. You are supposed to know that a geometric series with
   - $A(\neq 0)$: the intial term, and
   - r: the ratio of the adjacent terms so that $a_{n+1} = a_n \cdot r$

$$\begin{cases} \text{converges to } \dfrac{A}{1-r} & \text{when } |r| < 1, \\ \text{diverges} & \text{when } |r| \geq 1. \end{cases}$$

Warning: Let's look at the case where $|r| < 1$ and hence the geometric series converges. When you write the general term of the geometric series as

$$a_n = a \cdot r^{n-1}$$

and when the index starts with $n = 1$, we compute the initial term to be

$$A = a \cdot r^{1-1} = a$$

and hence

$$\sum_{n=1}^{\infty} a \cdot r^{n-1} = \frac{A}{1-r} = \frac{a}{1-r}.$$

However, when the index starts with $n = n_o \neq 1$, then the initial term is NOT equal to a and hence we have

$$\sum_{n=n_o}^{\infty} a \cdot r^{n-1} = \frac{A}{1-r} = \frac{a \cdot r^{n_o-1}}{1-r} \neq \frac{a}{1-r}.$$

Example Problems
- 10.3.76
- 10.3.77
- 10.3.81
- 10.3.83
- 10.3.86

2. You are supposed to be able to evaluate the telescoping series.

Example Problems
- 10.3.59
- 10.3.60
- 10.3.62
- 10.3.78
- 10.3.79

2. You are supposed to know how to detemine whether a given series is convergent or divergent, using the various tests below.

FAQ: Given a series, how do you know which test to use ?
Is there any easy algorithm or recipe to tell which test to use ?
Answer: NO ! Actually you are asking a "wrong" question.
Get rid of this high-school mentality (math = memorizing the recipe for a solution without understanding).

Given a series, what you should do is:
Step 1. Look for a series which is simpler yet similar to the given series.
Step 2. Determine whether that simpler series converges or diverges.
Step 3. The original series accordingly should converge or diverge. Justify your judgement by using an appropriate test.
Warning: There could be many tests to justify your answer in Step 3.

Summary of Tests:

(1) p**-series**

$$\sum \frac{1}{n^p} \begin{cases} \text{converges} & \text{when} \quad p > 1 \\ \text{diverges} & \text{when} \quad p \le 1 \end{cases}$$

(2) **Geometric seies**

$$\sum a \cdot r^{n-1} \ (a \ne 0) \begin{cases} \text{converges} & \text{when} \quad |r| < 1 \\ \text{diverges} & \text{when} \quad |r| \ge 1 \end{cases}$$

(3) **Comparison Test**

1st form: $0 \le a_n \le b_n, \sum b_n$ converges $\Longrightarrow \sum a_n$ converges.
2nd form: $0 \le b_n \le a_n, \sum b_n$ diverges $\Longrightarrow \sum a_n$ diverges.

(4) **Limit Comparison Test**

$$0 < a_n, b_n \text{ and } \lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$$
$$\Longrightarrow \sum a_n \& \sum b_n \text{ share the same destiny.}$$

Note: When $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$ or ∞, we cannot conclude $\sum a_n$ and $\sum b_n$ share the same destiny in general. The exceptions are:

① $0 < a_n, b_n$ and $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0, \sum b_n$ converges
$\Longrightarrow \sum a_n$ also converges.

② $0 < a_n, b_n$ and $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \infty, \sum b_n$ diverges
$\Longrightarrow \sum a_n$ also diverges.

(5) **Test for Divergence**

$\lim_{n\to\infty} a_n \ne 0 \Longrightarrow \sum a_n$ diverges.
Warning: There is NO (Test for Convergence), which says
$\lim_{n\to\infty} a_n = 0 \Longrightarrow \sum a_n$ converges.

(6) **Alternating Series Test**

$\sum a_n = \sum (-1)^{n-1 \text{ or } n} b_n$ with $b_n > 0$

converges if one checks the two conditions

$$\begin{cases} \text{(i)} & b_n \geq b_{n+1}, \\ \text{(ii)} & \lim_{n\to\infty} b_n = 0. \end{cases}$$

Note: If condition (ii) fails, i.e., if $\lim_{n\to\infty} b_n \neq 0$, then

$\lim_{n\to\infty} b_n \neq 0 \Longrightarrow \lim_{n\to\infty} a_n \neq 0 \Longrightarrow \sum a_n$ diverges.

Meanwhile, the failure of condition (i) does not necessarily imply that $\sum a_n$ diverges.

(7) **Absolute Convergence**

$\sum |a_n|$ converges $\Longrightarrow \sum a_n$ converges.

When this happens, we say $\sum a_n$ absolutely converges.

Note: When $\sum a_n$ converges even though $\sum |a_n|$ diverges, we say $\sum a_n$ conditionally converges.

(8) **Ratio Test**

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \begin{cases} < 1 & \Longrightarrow & \sum a_n \text{ (abs.) converges} \\ > 1 & \Longrightarrow & \sum a_n \text{ diverges} \\ = 1 & & \text{inconclusive} \end{cases}$$

(9) **Root Test**

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \begin{cases} < 1 & \Longrightarrow & \sum a_n \text{ (abs.) converges} \\ > 1 & \Longrightarrow & \sum a_n \text{ diverges} \\ = 1 & & \text{inconclusive} \end{cases}$$

(10) **Integral Test**

$a_n = f(n)$ where f is a continuous, positive, and decreasing function on $[1, \infty)$, then $\int_1^\infty f(x)dx$ and $\sum a_n$ share the same destiny.

Example Problems

- 10.4.12
- 10.4.16
- 10.4.20
- 10.4.22
- 10.5.22
- 10.5.23
- 10.5.24
- 10.5.31
- 10.6.52
- 10.6.53
- 10.6.63
- 10.7.21
- 10.7.23
- 10.7.44

3. You are supposed to be able to compute the n-th order Taylor polynomial

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

of a given function $f(x)$ centered at a. You are also supposed to know Taylor's theorem on how to express the remainder

$$R_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

where c is some number between x and a, and its estimate.

$$|R_n(x)| = |f(x) - p_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

where M is a number such that $|f^{(n+1)}(c)| \leq M$ for all c between x and a.

Example Problems

- 11.1.17
- 11.1.18
- 11.1.21
- 11.1.29

4. Given a power series, you are supposed to know how to determine the radius of convergence by first using the Ratio Test, and then to determine the interval of convergence by checking th behavior at the boundary points.

Example Problems

- 11.2.13
- 11.2.14
- 11.2.23

5. You are supposed to know how to find the power series expression for a function, using the basic formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

You are also supposed to know, given a poer series, how to find the power series expression for its derivative and integration.

Example Problems

- 11.2.46
- 11.2.55
- 11.2.56

6. Given a function, you are supposed to be able to find its Maclaurin series (power series centered at 0) and Taylor series (power series centered at a) by the basic formulas

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n. \end{aligned}$$

Example Problems

- 11.3.16
- 11.3.17
- 11.3.20