QUASI-PERFECT SCHEME-MAPS AND BOUNDEDNESS OF THE TWISTED INVERSE IMAGE FUNCTOR

JOSEPH LIPMAN AND AMNON NEEMAN

To Phillip Griffith, on his 65th birthday

Abstract. For a map $f\colon X \to Y$ of quasi-compact quasi-separated schemes, we discuss *quasi-perfection*, i.e., the right adjoint $f^\times$ of $\mathbf{R}f_*$ respects small direct sums. This is equivalent to the existence of a functorial isomorphism $f^\times\mathcal{O}_Y \otimes^{\mathbf{L}} \mathbf{L}f^*(-) \xrightarrow{\sim} f^\times(-)$; to *quasi-properness* (preservation by $\mathbf{R}f_*$ of pseudo-coherence, or just *properness* in the noetherian case) plus boundedness of $\mathbf{L}f^*$ (finite tor-dimensionality), or of the functor $f^\times$; and to some other conditions. We use a globalization, previously known only for divisorial schemes, of the local definition of pseudo-coherence of complexes, as well as a refinement of the known fact that the derived category of complexes with quasi-coherent homology is generated by a single perfect complex.

1. Introduction

This paper, inspired by [V, p. 396, Lemma 1 and Corollary 2], deals with matters raised there, but not yet fully treated in the literature.

Throughout, *scheme* will mean *quasi-compact quasi-separated scheme* (see [GD, §6.1, p. 290ff]), though weaker assumptions would sometimes suffice. Unless otherwise indicated, a *map* $f\colon X \to Y$ will be a *scheme-morphism,* necessarily quasi-compact and quasi-separated.

For a scheme X, $\mathbf{D}(X)$ is the (unbounded) derived category of the category of (sheaves of) $\mathcal{O}_X$-modules, and $\mathbf{D}_{\mathsf{qc}}(X)$ is the full subcategory whose objects are the $\mathcal{O}_X$-complexes whose homology sheaves are all quasi-coherent. For any map $f\colon X \to Y$, the derived functor $\mathbf{R}f_*\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ takes $\mathbf{D}_{\mathsf{qc}}(X)$ to $\mathbf{D}_{\mathsf{qc}}(Y)$ [Lp, Prop. (3.9.2)]. Grothendieck Duality theory asserts, to begin, that *the restriction* $\mathbf{R}f_*\colon \mathbf{D}_{\mathsf{qc}}(X) \to \mathbf{D}_{\mathsf{qc}}(Y)$ *has a right adjoint* $f^\times$, the "twisted inverse image functor" in our title.[1]

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[1] Warning: for nonproper maps of noetherian schemes, the usual twisted inverse image $f^!$ differs from $f^\times$, and is not covered by this paper. For that, see, e.g., [Lp, §4.9].

A proof for maps of *separated* schemes, suggested by Deligne's appendix to [H], is described in [Lp, §4.1]. This proof depends ultimately on the Special Adjoint Functor Theorem, applied to categories of sheaves. A more direct approach, via Brown Representability—which applies immediately to derived categories—is given in [N1]. Originally this too required separability, but now that assumption can be dropped because of [BB, p. 9, Thm. 3.3.1], which gives that $\mathbf{D}_{\mathsf{qc}}(X)$ is compactly generated, and because $\mathbf{R}f_*$ commutes with $\mathbf{D}_{\mathsf{qc}}$-coproducts (= direct sums) [Lp, (3.9.3.3)].[2]

The functor $f^\times$ emerging from these proofs commutes with translation (=suspension) of complexes, and is *bounded-below* (*way-out right* in the sense of [H, p. 68]), i.e., there exists an integer m such that for every $F \in \mathbf{D}_{\mathsf{qc}}(Y)$ with $H^iF = 0$ for all i less than some integer $n(F)$, it holds that $H^i f^\times F = 0$ for all $i < n(F) - m$ (see [Lp, (4.1.8) and the remarks preceding it]).

"Bounded-below" has a similar meaning for any functor between derived categories. *Bounded-above* is defined in an analogous way, with $>$ (resp. $+$) in place of $<$ (resp. $-$). A functor is *bounded* if it is bounded both above and below. Boundedness enables a potent form of induction in derived categories, expressed by the "way-out Lemmas" [H, p. 68, Prop. 7.1 and p. 73, Prop. 7.3].

For example, the left adjoint $\mathbf{L}f^*$ of $\mathbf{R}f_*$ is always bounded-above; and $\mathbf{L}f^*$ is bounded iff f has *finite tor-dimension* (a.k.a. *finite flat dimension*), that is, there is an integer $d \geq 0$ such that for each $x \in X$ there exists an exact sequence of $\mathcal{O}_{Y,f(x)}$-modules

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to \mathcal{O}_{X,x} \to 0$$

with P_i flat over $\mathcal{O}_{Y,f(x)}$ $(0 \leq i \leq d)$.

We will be concerned with the relation between boundedness of the right adjoint $f^\times$ and the left adjoint $\mathbf{L}f^*$, especially in the context of *quasi-perfection,* a property of maps to be discussed at length now and in §2.

Definition 1.1. We say a map $f\colon X \to Y$ is *quasi-perfect* if $f^\times$ respects direct sums in $\mathbf{D}_{\mathsf{qc}}$, i.e., for any small $\mathbf{D}_{\mathsf{qc}}(Y)$-family (E_α) the natural map is an isomorphism

$$\bigoplus_\alpha f^\times E_\alpha \xrightarrow{\ \sim\ } f^\times\Big(\bigoplus_\alpha E_\alpha\Big).$$

As will be explained below, quasi-perfection is also characterized by the existence of a canonical isomorphism

$$f^\times\mathcal{O}_Y \otimes^{\mathbf{L}} \mathbf{L}f^*F \xrightarrow{\ \sim\ } f^\times F \qquad \big(F \in \mathbf{D}_{\mathsf{qc}}(Y)\big).$$

More characterizations are given in §2—for instance, via compatibility of $f^\times$ with tor-independent base change (Theorem 2.7). That section also brings in the related condition on maps of being *perfect*, i.e., pseudo-coherent

[2]Subsequently, a slightly simpler proof was given in [BB, p. 14, 3.3.4]. (In that proof one needs to replace "flabby" by "quasi-flabby," see [Kf, §2].)

and of finite tor-dimension. (Pseudo-coherence will be reviewed in §2. It holds, for instance, for all finite-type maps of noetherian schemes; and then descent to the noetherian case yields that every flat, finitely presentable map is pseudo-coherent.) For example, for a proper map f of noetherian schemes, *f is quasi-perfect $\Leftrightarrow$ f is perfect $\Leftrightarrow$ $f^\times$ is bounded.*

It is stated in [V, p. 396, Lemma 1] that any proper map f of finite-dimensional noetherian schemes is quasi-perfect. In general, however, this fails even for closed immersions. But $f^\times$ *does* respect direct sums when the summands E_α are uniformly homologically bounded below, i.e., there exists an integer n such that for all α, $H^iE_\alpha = 0$ whenever $i < n$ [Lp, (4.7.6)(b)]. Consequently, *if the functor $f^\times$ is bounded, then f is quasi-perfect.*

Our main results say more. But first, call a map $f\colon X \to Y$ *quasi-proper* if $\mathbf{R}f_*$ takes pseudo-coherent $\mathcal{O}_X$-complexes to pseudo-coherent $\mathcal{O}_Y$-complexes. (Again, pseudo-coherence is explained in §2. In particular, if X is noetherian then $E \in \mathbf{D}(X)$ is pseudo-coherent iff the homology sheaves $H^n(E)$ are all coherent, and vanish for $n \gg 0$.) Kiehl showed that *every proper pseudo-coherent map is quasi-proper.* Consequently, any flat, finitely presentable, proper map, being pseudocoherent, is quasi-proper (and perfect and quasi-perfect as well). Moreover, *when Y is noetherian, every finite-type separated quasi-proper $f\colon X \to Y$ is proper.*

Here are the main results.

Theorem 1.2. *For a map $f\colon X \to Y$, the following are equivalent*:

(i) *f is quasi-perfect* (*resp. perfect*).

(ii) *f is quasi-proper* (*resp. pseudo-coherent*) *and has finite tor-dimension.*

(iii) *f is quasi-proper* (*resp. pseudo-coherent*) *and $f^\times$ is bounded.*

Hence, by Kiehl's theorem, *every proper perfect map is quasi-perfect.*

The implication (i) $\Rightarrow$ (iii) is worked out in §4. The proofs in §4 are based on Theorems 4.1 and 4.2, which are of independent interest.

Theorem 4.1 states that for a scheme X, any pseudo-coherent $\mathcal{O}_X$-complex can be "arbitrarily-well approximated," *globally,* by a perfect complex. (*Local* approximability is essentially the definition of pseudo-coherence. The global result was previously known only for divisorial schemes.)

This leads to quasi-proper maps being characterized as those f such that $\mathbf{R}f_*$ takes perfect complexes to pseudo-coherent ones. Since by Prop. 2.1, quasi-perfect maps are those f such that $\mathbf{R}f_*$ takes perfect complexes to perfect ones, it follows at once that quasi-perfect maps are quasi-proper.

Theorem 4.2 refines a theorem of Bondal and van den Bergh [BB, p. 9, Thm. 3.1.1] which states that the triangulated category $\mathbf{D}_{\mathsf{qc}}(X)$ is generated by a single perfect complex. With this in hand, one can prove Corollary 4.3.1, which says that for any quasi-perfect *or* perfect f as above, $f^\times$ is bounded.

The implication (iii) $\Rightarrow$ (ii) results from Theorem 3.1, which says, for any $f\colon X \to Y$ as above, *if $f^\times$ is bounded then f has finite tor-dimension.*

Finally, the implication (ii) ⇒ (i) holds by definition for the resp. case, and is proved for the other case in §2, Example 2.2(a).

Let us call a map $f\colon X \to Y$ *locally embeddable* if every $y \in Y$ has an open neighborhood V over which the induced map $f^{-1}V \to V$ factors as $f^{-1}V \xrightarrow{i} Z \xrightarrow{p} V$ where i is a closed immersion and p is smooth. (For instance, any quasi-projective f satisfies this condition.) Proposition 2.5 asserts that *any quasi-proper locally embeddable map is pseudo-coherent.* A similar proof shows that any quasi-perfect locally embeddable map is perfect. By 1.2, then, *a locally embeddable map is quasi-perfect iff it is quasi-proper and perfect.*

The equivalence of (i) and (ii) in Theorem 1.2 generalizes [V, p. 396, Cor. 2], in view of the following characterization (mentioned above) of quasi-perfection.

For a map $f\colon X \to Y$, and for any $E \in \mathbf{D}_{\mathsf{qc}}(X)$, $F \in \mathbf{D}_{\mathsf{qc}}(Y)$, with $\underline{\underline{\otimes}} := \otimes^{\mathbf{L}}$, the derived tensor product, the "projection map"

$$\pi\colon (\mathbf{R}f_*E) \underline{\underline{\otimes}} F \to \mathbf{R}f_*(E \underline{\underline{\otimes}} \mathbf{L}f^*F),$$

defined to be adjoint to the natural composite map

$$\mathbf{L}f^*\big((\mathbf{R}f_*E) \underline{\underline{\otimes}} F\big) \xrightarrow{\;\sim\;} (\mathbf{L}f^*\mathbf{R}f_*E) \underline{\underline{\otimes}} \mathbf{L}f^*F \to E \underline{\underline{\otimes}} \mathbf{L}f^*F,$$

is an *isomorphism.* (This is well-known under more restrictive hypotheses; for a proof in the stated generality, see [Lp, Prop. (3.9.4)].) There results a natural map

$$\chi_F\colon f^\times\mathcal{O}_Y \underline{\underline{\otimes}} \mathbf{L}f^*F \to f^\times F \qquad \big(F \in \mathbf{D}_{\mathsf{qc}}(Y)\big), \tag{1.3}$$

adjoint to the natural composite map

$$\mathbf{R}f_*(f^\times\mathcal{O}_Y \underline{\underline{\otimes}} \mathbf{L}f^*F) \underset{\pi^{-1}}{\xrightarrow{\;\sim\;}} \mathbf{R}f_*f^\times\mathcal{O}_Y \underline{\underline{\otimes}} F \to \mathcal{O}_Y \underline{\underline{\otimes}} F = F.$$

It is clear (since $\underline{\underline{\otimes}}$ and $\mathbf{L}f^*$ both respect direct sums, see e.g., [Lp, 3.8.2]) that *if χ_F is an isomorphism for all $F \in \mathbf{D}_{\mathsf{qc}}(Y)$ then f is quasi-perfect;* and Proposition 2.1 gives the converse.

2. Quasi-perfect maps

For surveying quasi-perfection in more detail, starting with Proposition 2.1, we need some preliminaries.

First, a brief review of the notion of pseudo-coherence of complexes. (Details can be found in the primary source [I, Exposé III], or, perhaps more accessibly, in [TT, pp. 283*ff,* §2]; a summary appears in [Lp, §4.3].) The idea is built up from that of *strictly perfect* $\mathcal{O}_X$-complex, i.e., bounded complex of finite-rank free $\mathcal{O}_X$-modules.

For $n \in \mathbb{Z}$, a map $\xi\colon P \to E$ in $\mathbf{K}(X)$, the homotopy category of $\mathcal{O}_X$-complexes, (resp. in $\mathbf{D}(X)$), is said to be an *n-quasi-isomorphism* (resp. *n-isomorphism*) if the following two equivalent conditions hold:

(1) The homology map $H^j(\xi)\colon H^j(P) \to H^j(E)$ is bijective for all $j > n$ and surjective for $j = n$.

(2) For any $\mathbf{K}(X)$- (resp. $\mathbf{D}(X)$-)triangle

$$P \xrightarrow{\ \xi\ } E \longrightarrow Q \longrightarrow P[1],$$

it holds that $H^j(Q) = 0$ for all $j \ge n$.

Then E is said to be *n-pseudo-coherent* if X has an open covering (U_α) such that for each α there exists a strictly perfect $\mathcal{O}_{U_\alpha}$-complex P_α and an n-quasi-isomorphism (or equivalently, an n-isomorphism) $P_\alpha \to E|_{U_\alpha}$, see [I, p. 98, Définition 2.3]; and E is *pseudo-coherent* if E is n-pseudo-coherent for every n. If $\mathcal{O}_X$ is coherent, this means simply that F has coherent homology sheaves, vanishing in all sufficiently large degrees [*ibid.,* p. 116, top]. When X is noetherian and finite-dimensional, it means that F is globally $\mathbf{D}$-isomorphic to a bounded-above complex of coherent $\mathcal{O}_X$-modules [*ibid.,* p. 168, Cor. 2.2.2.1].

A complex $E \in \mathbf{D}(X)$ (X a scheme) is said to be *perfect* if it is locally $\mathbf{D}$-isomorphic to a strictly perfect $\mathcal{O}_X$-complex. More precisely, E is said to have *perfect amplitude in* $[a, b]$ ($a \le b \in \mathbb{Z}$) if locally on X, E is $\mathbf{D}$-isomorphic to a bounded complex of finite-rank free $\mathcal{O}_X$-modules vanishing in all degrees $< a$ or $> b$. Thus E is perfect iff it has perfect amplitude in some interval $[a, b]$.

By [I, p. 134, 5.8], E has perfect amplitude in $[a, b]$ iff E is $(a-1)$-pseudo-coherent and has tor-amplitude in $[a, b]$ (i.e., is globally $\mathbf{D}$-isomorphic to a flat complex vanishing in all degrees $< a$ and $> b$). So *E is perfect iff it is pseudo-coherent and has finite tor-dimension* (the latter meaning that it is $\mathbf{D}$-isomorphic to a bounded flat complex).

A map $f\colon X \to Y$ is *pseudo-coherent* if every $x \in X$ has an open neighborhood U such that the restriction $f|_U$ factors as $U \xrightarrow{i} Z \xrightarrow{p} Y$, where i is a closed immersion such that $i_*\mathcal{O}_U$ is pseudo-coherent on Z, and p is smooth [I, p. 228, Défn. 1.2]. Pseudo-coherent maps are finitely presentable. Compositions of pseudo-coherent maps are pseudo-coherent [I, p. 236, Cor. 1.14].

A map is *perfect* if it is pseudo-coherent and has finite tor-dimension [I, p. 250, Défn. 4.1]. Any smooth map is perfect, any regular immersion (= closed immersion corresponding to a quasi-coherent ideal generated locally by a regular sequence) is perfect, and compositions of perfect maps are perfect [I, p. 253, Cor. 4.5.1(a)].

For noetherian Y, any finite-type $f\colon X \to Y$ is pseudo-coherent. Pseudo-coherence (resp. perfection) of maps survives tor-independent base change [I, p. 233, Cor. 1.10; p. 257, Cor. 4.7.2]. Hence, by descent to the noetherian case [EGA, IV, (11.2.7)], *every flat finitely-presentable map is perfect.*

A map $f\colon X \to Y$ is *quasi-proper* if $\mathbf{R}f_*$ takes pseudo-coherent $\mathcal{O}_X$-complexes to pseudo-coherent $\mathcal{O}_Y$-complexes.

Kiehl's Finiteness Theorem [Kl, p. 315, Thm. 2.9′] (first proved by Illusie for projective maps [I, p. 236, Thm. 2.2]) generalizes preservation of coherence by higher direct images under proper maps of noetherian schemes. It states that *every proper pseudo-coherent map is quasi-proper.*

This theorem (or its special case [I, p. 240, Cor. 2.5]), plus [Lp, Ex. (4.3.9)]) implies that *if Y is noetherian then a finite-type separated $f\colon X \to Y$ is quasi-proper iff it is proper.*

For details in the proof of the following Proposition, and for some subsequent considerations, recall that an object C in a triangulated category $\mathcal{T}$ is *compact* if for every small $\mathcal{T}$-family (E_α) the natural map is an isomorphism

$$\bigoplus_\alpha \operatorname{Hom}(C, E_\alpha) \xrightarrow{\sim} \operatorname{Hom}(C, \bigoplus_\alpha E_\alpha).$$

For any scheme X, the compact objects of $\mathbf{D}_{\mathsf{qc}}(X)$ are just the perfect complexes, of which one is a generator [BB, p. 9, Thm. 3.1.1].

Proposition 2.1. *For a map $f\colon X \to Y$, the following are equivalent:*

(i) *f is quasi-perfect* (*Definition* 1.1).

(ii) *The functor $\mathbf{R}f_*$ takes perfect complexes to perfect complexes.*

(ii)$'$ *If S is a perfect generator of $\mathbf{D}_{\mathsf{qc}}(X)$ then $\mathbf{R}f_*S$ is perfect.*

(iii) *The twisted inverse image functor $f^\times$ has a right adjoint.*

(iv) *For all $F \in \mathbf{D}_{\mathsf{qc}}(Y)$, the map in* (1.3) *is an* isomorphism

$$\chi_F\colon f^\times\mathcal{O}_Y \underset{=}{\otimes} \mathbf{L}f^*F \xrightarrow{\ \sim\ } f^\times F.$$

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (ii)$'$: [N1, p. 224, Thm. 5.1].

(i) $\Rightarrow$ (iii): [N1, p. 223, Thm. 4.1].

(iii) $\Rightarrow$ (i): simple.

(i) $\Rightarrow$ (iv) $\Rightarrow$ (i): See [N1, p. 226, Thm. 5.4].

To be precise, the results in [N1] are proved for *separated* schemes; but with the remark preceding Prop. 2.1, one readily verifies that the proofs survive without any separability requirement. □

Examples 2.2. (a) *Any quasi-proper map f of finite tor-dimension—in particular, by Kiehl's theorem, any proper perfect map—is quasi-perfect.* Indeed, $\mathbf{R}f_*$ preserves pseudo-coherence, and by [I, p. 250, 3.7.2] (a consequence of the projection isomorphism mentioned near the end of the above Introduction), $\mathbf{R}f_*$ preserves finite tor-dimensionality of complexes; so Prop. 2.1(ii) holds.

(b) Let $f\colon X \to Y$ be a map with X *divisorial*—i.e., X has an ample family $(\mathcal{L}_i)_{i\in I}$ of invertible $\mathcal{O}_X$-modules [I, p. 171, Défn. 2.2.5]. Then [N1, p. 212, Example 1.11 and p. 224, Theorem 5.1] show that *f is quasi-perfect $\Leftrightarrow$ for each $i \in I$, the $\mathcal{O}_Y$-complex $\mathbf{R}f_*(\mathcal{L}_i^{\otimes -n_i})$ is perfect for all $n_i \gg 0$.*

(c) Let f be quasi-projective and let $\mathcal{L}$ be an f-ample invertible sheaf. Then *f is quasi-perfect $\Leftrightarrow$ the $\mathcal{O}_Y$-complex $\mathbf{R}f_*(\mathcal{L}^{\otimes -n})$ is perfect for all $n \gg 0$.*

Indeed, condition (ii) in Prop. 2.1, together with the compatibility of $\mathbf{R}f_*$ and open base change, implies that quasi-perfection is a property of f which can be checked locally on Y, and the same holds for perfection of $\mathbf{R}f_*(\mathcal{L}^{\otimes -n})$; so we may assume Y affine, and apply (b).

(d) For a *finite* map $f\colon X \to Y$ the following are equivalent:
(i) f is quasi-perfect.
(ii) f is perfect.
(iii) The complex $f_*\mathcal{O}_X \cong \mathbf{R}f_*\mathcal{O}_X$ is perfect.
This follows quickly from (a) and from Proposition 2.1(ii).

A *tor-independent square* is a fiber square of maps

$$\begin{array}{ccc} X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ u\ ]{} & Y \end{array} \tag{2.3}$$

(that is, the natural map is an isomorphism $X' \xrightarrow{\sim} X \times_Y Y'$) such that for all $x \in X$, $y' \in Y'$ and $y \in Y$ with $f(x) = u(y') = y$, and all $i > 0$, $\mathrm{Tor}_i^{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y'}) = 0$.

The following stability properties will be useful.

Proposition 2.4. *For any tor-independent square* (2.3),
(i) *If the functor $f^\times$ is bounded then so is $g^\times$.*
(ii) *If f is quasi-perfect then so is g.*
(iii) *If f is quasi-proper then so is g.*

Proof. (i) and (ii) are proved in [Lp, (4.7.3.1)]; and (iii) is treated in Prop. 4.4 below (a slight change in whose proof gives another proof of (ii)). □

Since perfection (resp. pseudo-coherence) is a local property of complexes, and $\mathbf{R}f_*$ is compatible with open base change on Y, we deduce:

Corollary 2.4.1. *Let $f\colon X \to Y$ be a map, and let $(Y_i)_{i\in I}$ be an open cover of Y. Then f is quasi-perfect* (*resp. quasi-proper*) $\Leftrightarrow$ *for all i, the same is true of the induced map $f^{-1}Y_i \to Y_i$.*

Proposition 2.5. *Let $f\colon X \to Y$ be a* locally embeddable *map, i.e., every $y \in Y$ has an open neighborhood V over which the induced map $f^{-1}V \to V$ factors as $f^{-1}V \xrightarrow{i} Z \xrightarrow{p} V$ where i is a closed immersion and p is smooth.* (*For instance, any quasi-projective f satisfies this condition* [EGA, II, (5.3.3)].)
(i) *If f is quasi-proper then f is pseudo-coherent.*
(ii) *If f is quasi-perfect then f is perfect.*

Proof. By Corollary 2.4.1, quasi-properness (resp. quasi-perfection) of f is a property local over Y; and since they are compatible with tor-independent base change, the same is true of pseudo-coherence and perfection. So we may as well assume that $X = f^{-1}V$. Then it suffices to show that the complex $i_*\mathcal{O}_X$ is pseudo-coherent when f is quasi-proper, (resp., by [I, p. 252, Prop. 4.4], that $i_*\mathcal{O}_X$ is perfect when f is quasi-perfect).

But i factors as $X \xrightarrow{\gamma} X \times_Y Z \xrightarrow{g} Z$ with γ the graph of i and g the projection. The map γ is a local complete intersection [EGA, IV, (17.12.3)], so the complex $\gamma_*\mathcal{O}_X$ is perfect. Also, g arises from f by flat base change, so by Proposition 2.4, g is quasi-proper (resp. quasi-perfect). Hence $i_*\mathcal{O}_X = \mathbf{R}i_*\mathcal{O}_X = \mathbf{R}g_*\gamma_*\mathcal{O}_X$ is indeed pseudo-coherent (resp. perfect). $\square$

(2.6). For any tor-independent square (2.3), the map

(2.6.1) $$\theta(E)\colon \mathbf{L}u^*\mathbf{R}f_*E \to \mathbf{R}g_*\mathbf{L}v^*E \qquad \big(E \in \mathbf{D}_{\mathsf{qc}}(X)\big)$$

adjoint to the natural composition

$$\mathbf{R}f_*E \to \mathbf{R}f_*\mathbf{R}v_*\mathbf{L}v^*E \cong \mathbf{R}u_*\mathbf{R}g_*\mathbf{L}v^*E$$

(equivalently, to $\mathbf{L}g^*\mathbf{L}u^*\mathbf{R}f_*E \cong \mathbf{L}v^*\mathbf{L}f^*\mathbf{R}f_*E \to \mathbf{L}v^*E$) is an isomorphism, so that one has a *base-change map*

(2.6.2) $$\beta(F)\colon \mathbf{L}v^*f^\times F \to g^\times\mathbf{L}u^*F \qquad \big(F \in \mathbf{D}_{\mathsf{qc}}(Y)\big)$$

adjoint to the natural composition

$$\mathbf{R}g_*\mathbf{L}v^*f^\times F \xrightarrow[\theta^{-1}]{\sim} \mathbf{L}u^*\mathbf{R}f_*f^\times F \to \mathbf{L}u^*F.$$

The fundamental *independent base-change* theorem states that:

Let there be given a tor-independent square (2.3) *and an* $F \in \mathbf{D}_{\mathsf{qc}}(Y)$*. If* f *is quasi-proper,* u *has finite tor-dimension, and* $H^nF = 0$ *for all* $n \ll 0$*, then* $\beta(F)$ *is an isomorphism.*

This theorem is well-known, at least under more restrictive hypotheses. For a treatment in full generality, see [Lp, §§4.4–4.6].

One consequence, in view of Proposition 2.4(i), is:

Corollary 2.6.3. *Let* $f\colon X \to Y$ *be a quasi-proper map and let* $(Y_i)_{i\in I}$ *be an open cover of* Y*. Then* $f^\times$ *is bounded* $\Leftrightarrow$ *for all* i*, the same is true of the induced map* $f^{-1}Y_i \to Y_i$*.*

For quasi-perfect f, a stronger base-change theorem holds—which, together with boundedness of $f^\times$ (Corollary 4.3.1), characterizes quasi-perfection:

Theorem 2.7 ([Lp, Thm. 4.7.4])**.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ u\ ]{} & Y \end{array}$$

be a tor-independent square, with f *quasi-perfect. Then for all* $F \in \mathbf{D}_{\mathsf{qc}}(Y)$ *the base-change map of* (2.6.2) *is an* isomorphism

$$\beta(F)\colon v^*f^\times F \xrightarrow{\ \sim\ } g^\times u^*F.$$

The same holds, with no assumption on f*, whenever* u *is finite and perfect.*

Conversely, the following conditions on a map $f\colon X\to Y$ are equivalent; and if Y is separated and $f^\times$ bounded above, they imply that f is quasi-perfect:

(i) *For any flat affine universally bicontinuous map $u\colon Y'\to Y$,*[3] *the base-change map associated to the (tor-independent) square*

$$\begin{array}{ccc} Y'\times_Y X = X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ u\ ]{} & Y \end{array}$$

is an isomorphism

$$\beta(\mathcal{O}_Y)\colon v^*f^\times\mathcal{O}_Y \xrightarrow{\ \sim\ } g^\times u^*\mathcal{O}_Y.$$

(ii) *The map in* (1.3) *is an isomorphism*

$$\chi_F\colon f^\times\mathcal{O}_Y \underset{=}{\otimes} \mathbf{L}f^*F \xrightarrow{\ \sim\ } f^\times F$$

whenever F is a flat quasi-coherent $\mathcal{O}_Y$-module.

Keeping in mind Corollary 4.3.1 below (f quasi-perfect $\Rightarrow f^\times$ bounded), we can deduce:

Corollary 2.7.1. *When Y is separated, a map $f\colon X\to Y$ is quasi-perfect iff $f^\times$ is bounded and the following two conditions hold*:

(i) *If $u\colon Y'\to Y$ is an open immersion, and if $v\colon Y'\times_Y X\to X$ and $g\colon Y'\times_Y X\to Y$ are the projection maps, then the base-change map is an isomorphism*

$$\beta(\mathcal{O}_Y)\colon v^*f^\times\mathcal{O}_Y \xrightarrow{\ \sim\ } g^\times u^*\mathcal{O}_Y.$$

Equivalently (*see* [Lp, §4.6, subsection V]), *for all $E\in\mathbf{D}_{\mathsf{qc}}(X)$ the natural composite map*

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(E,f^\times\mathcal{O}_Y)\to\mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*E,\mathbf{R}f_*f^\times\mathcal{O}_Y)\to\mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*E,\mathcal{O}_Y)$$

is an isomorphism.

(ii) *If (F_α) is a filtered direct system of flat quasi-coherent $\mathcal{O}_Y$-modules, then for all $n\in\mathbb{Z}$ the natural map is an isomorphism*

$$\varinjlim_\alpha H^n(f^\times F_\alpha) \xrightarrow{\ \sim\ } H^n(f^\times \varinjlim_\alpha F_\alpha).$$

Remarks. 1. Conditions (i) and (ii) in Theorem 2.7 are connected via the flat, affine, and universally bicontinuous natural map $\operatorname{Spec}(S_{\le 1}(F))\to Y$, where $S_{\le 1}(F)$ is the $\mathcal{O}_Y$-algebra $\mathcal{O}_Y\oplus F$ with $F^2=0$.

2. The idea behind the proof of Corollary 2.7.1 is to use Lazard's theorem that over a commutative ring A any flat module is a $\varinjlim$ of finite-rank free A-modules [GD, p. 163, (6.6.24)], to show that (i) and (ii) imply condition (ii) in Theorem 2.7.

[3] *Universally bicontinuous* means that for any $Y''\to Y$ the resulting projection map $Y'\times_Y Y''\to Y''$ is a homeomorphism onto its image [GD, p. 249, Défn. (3.8.1)].

3. Boundedness of $f^\times$ implies finite tor-dimension

Theorem 3.1. *Let $f\colon X \to Y$ be a map. If $f^\times$ is bounded then f has finite tor-dimension.*

The proof uses the following two Lemmas.

An $\mathcal{O}_X$-complex E is *a_locally projective* ($a \in \mathbb{Z}$) if there is a $b \geq a$ and an affine open covering $\big(U_i := \mathrm{Spec}(A_i)\big)_{i\in I}$ of X such that for each $i \in I$, the restriction $E|_{U_i}$ is $\mathbf{D}$-isomorphic to a quasi-coherent direct summand of a complex F of free $\mathcal{O}_{U_i}$-modules, with F vanishing in all degrees outside $[a, b]$.

Every complex with perfect amplitude in $[a, b]$ (§2) is a_locally projective.

Lemma 3.2. *For any scheme X, there is an integer $s > 0$ such that for all $a \in \mathbb{Z}$ and a_locally projective $E \in \mathbf{D}(X)$, if $G \in \mathbf{D}_{\mathsf{qc}}(X)$ and $H^jG = 0$ for all $j > a - s$ then* $\mathrm{Hom}_{\mathbf{D}(X)}(E, G) = 0$.

Lemma 3.3. *Let $f\colon X \to Y$ be a perfect map, of tor-dim $d < \infty$. Then there exists an integer $t > 0$ such that for any a_locally projective $E \in \mathbf{D}_{\mathsf{qc}}(X)$, $\mathbf{R}f_*E \in \mathbf{D}_{\mathsf{qc}}(Y)$ is $(a - d - t)$_locally projective.*

These Lemmas are proved below.

Proof of Theorem 3.1.

Part (i) of Proposition 2.4 gives an immediate reduction to the case where Y is affine, say $Y = \mathrm{Spec}(A)$. We need to show in this case that for any open immersion $\iota\colon U \hookrightarrow X$ with U affine the $\mathcal{O}_Y$-module $f_*\iota_*\mathcal{O}_U$ has finite tor-dimension.

Since U is affine, there are natural isomorphisms

$$f_*\iota_*\mathcal{O}_U = (f\iota)_*\mathcal{O}_U \xrightarrow{\ \sim\ } \mathbf{R}(f\iota)_*\mathcal{O}_U \xrightarrow{\ \sim\ } \mathbf{R}f_*\mathbf{R}\iota_*\mathcal{O}_U.$$

So for any $G \in \mathbf{D}_{\mathsf{qc}}(Y)$ there are natural isomorphisms

$$\mathrm{Hom}_{\mathbf{D}(Y)}(f_*\iota_*\mathcal{O}_U, G) \cong \mathrm{Hom}_{\mathbf{D}(Y)}(\mathbf{R}f_*\mathbf{R}\iota_*\mathcal{O}_U, G) \cong \mathrm{Hom}_{\mathbf{D}(X)}(\mathbf{R}\iota_*\mathcal{O}_U, f^\times G).$$

Lemma 3.3 provides an integer t such that if U is *any* quasi-compact open subscheme of X, with inclusion $\iota\colon U \subset X$, then $\mathbf{R}\iota_*\mathcal{O}_U$ is $(-t)$_locally projective. By Lemma 3.2 and the boundedness of $f^\times$, it follows, for U affine, G a quasi-coherent $\mathcal{O}_Y$-module, and some $j \gg 0$ not depending on G, that

$$\begin{aligned}\mathrm{Ext}^j(f_*\iota_*\mathcal{O}_U, G) &= \mathrm{Hom}_{\mathbf{D}(Y)}(f_*\iota_*\mathcal{O}_U, G[j])\\ &\cong \mathrm{Hom}_{\mathbf{D}(X)}(\mathbf{R}\iota_*\mathcal{O}_U, f^\times G[j]) = 0.\end{aligned}$$

The natural equivalences $\mathbf{D}(A) \xrightarrow{\ \approx\ } \mathbf{D}(Y_{\mathsf{qc}}) \xrightarrow{\ \approx\ } \mathbf{D}_{\mathsf{qc}}(Y)$ (where Y_{qc} is the category of quasi-coherent $\mathcal{O}_Y$-modules—see [BN, p. 30, Cor. 5.5]) show then that $f_*\iota_*\mathcal{O}_U$ has a resolution by the sheafification of a bounded projective A-complex, and thus has finite tor-dimension, as desired. □

Proof of Lemma 3.2.

Let us call an open $U \subset X$ *E-good* if U is affine, say $U = \mathrm{Spec}(A)$, and if there is a $b \ge a$ such that the restriction $E|_U$ is $\mathbf{D}$-isomorphic to the sheafification of a projective A-complex E vanishing in all degrees outside $[a, b]$.

Clearly, every quasi-compact open subset of X is a finite union of E-good open subsets. Hence, as in the proof of [BB, p. 13, Prop. 3.3.1], it will suffice to show that Lemma 3.2 holds for X if X itself is E-good, or if $X = X_1 \cup X_2$ with X_1 and X_2 quasi-compact open subsets such that Lemma 3.2 holds for X_1, X_2 and $X_1 \cap X_2$ (which is also quasi-compact, since X is quasi-separated).

Suppose first that $X = \mathrm{Spec}(A)$ is E-good. Let E be as in the definition of E-good, and let $G \in \mathbf{D}_{\mathsf{qc}}(X)$ be such that $H^j G = 0$ for all $j > a - 1$. The natural equivalence of categories $\mathbf{D}(X_{\mathsf{qc}}) \xrightarrow{\approx} \mathbf{D}_{\mathsf{qc}}(X)$ (where X_{qc} is the category of quasi-coherent $\mathcal{O}_X$-modules) allows us to assume G quasi-coherent, so that G is the sheafification of an A-complex G; and further, after applying the well-known truncation functor (see e.g., [Lp, §1.10]) we can assume that G vanishes in all degrees $> a - 1$.

The dual versions of [Lp, (2.3.4) and (2.3.8)(v)], and the equivalences $\mathbf{D}(A) \xrightarrow{\approx} \mathbf{D}(X_{\mathsf{qc}})$, $\mathbf{D}(X_{\mathsf{qc}}) \xrightarrow{\approx} \mathbf{D}_{\mathsf{qc}}(X)$, yield natural isomorphisms, with $\mathbf{K}(A)$ the homotopy category of A-complexes:

$$\mathrm{Hom}_{\mathbf{K}(A)}(\mathrm{E}, \mathrm{G}) \cong \mathrm{Hom}_{\mathbf{D}(A)}(\mathrm{E}, \mathrm{G}) \cong \mathrm{Hom}_{\mathbf{D}(X_{\mathsf{qc}})}(E, G) \cong \mathrm{Hom}_{\mathbf{D}(X)}(E, G).$$

So since E vanishes in all degrees $< a$ and G vanishes in all degrees $> a - 1$, therefore $\mathrm{Hom}_{\mathbf{D}(X)}(E, G) = 0$, proving Lemma 3.2 in this case.

Suppose next that $X = X_1 \cup X_2$ as above. Let $s > 0$ be such that Lemma 3.2 holds with this s for all three of X_1, X_2, and $X_1 \cap X_2$. Let $G \in \mathbf{D}_{\mathsf{qc}}(X)$ satisfy $H^j G = 0$ for all $j > a - (s + 1)$. Let $i\colon X_1 \hookrightarrow X$, $j\colon X_2 \hookrightarrow X$, and $k\colon X_1 \cap X_2 \hookrightarrow X$ be the inclusion maps. One gets the natural triangle

$$G \longrightarrow \mathbf{R}i_*i^*G \oplus \mathbf{R}j_*j^*G \longrightarrow \mathbf{R}k_*k^*G \longrightarrow G[1],$$

by applying the usual exact sequence, holding for any flasque $\mathcal{O}_X$-module F,

$$0 \to F \to i_*i^*F \oplus j_*j^*F \to k_*k^*F \to 0$$

to an injective q-injective resolution[4] of E^+. There results an exact sequence [H, p, 21, 1.1(b)], with $\mathrm{Hom} := \mathrm{Hom}_{\mathbf{D}(X)}$,

$$\mathrm{Hom}(E, \mathbf{R}k_*k^*G[-1]) \to \mathrm{Hom}(E, G) \to \mathrm{Hom}(E, \mathbf{R}i_*i^*G) \oplus \mathrm{Hom}(E, \mathbf{R}j_*j^*G).$$

Adjointness of $\mathbf{R}k_*$ and $\mathbf{L}k^* = k^*$ gives that

$$\mathrm{Hom}_{\mathbf{D}(X)}(E, \mathbf{R}k_*k^*G[-1]) \cong \mathrm{Hom}_{\mathbf{D}(X_1 \cap X_2)}(k^*E, k^*G[-1]);$$

and Lemma 3.2 makes these groups vanish. Similarly, $\mathrm{Hom}(E, \mathbf{R}j_*j^*G) = 0$ and $\mathrm{Hom}(E, \mathbf{R}i_*i^*G) = 0$. Hence $\mathrm{Hom}(E, G) = 0$. $\square$

[4]Another term for "q-injective" is "K-injective"—see [Lp, (2.3.2.3), (2.3.5)].

Proof of Lemma 3.3.

The question is local on Y, so we may assume Y affine, say $Y = \mathrm{Spec}(B)$.

Arguing as in the preceding proof, suppose first that X is E-good. We begin with the case $E = \mathcal{O}_X$. Then for some $t > 0$, f factors as

$$X \xrightarrow{\iota} Y_t := \mathrm{Spec}(B[T_1, T_2, \dots, T_t]) \xrightarrow{\pi} \mathrm{Spec}(B),$$

where $T_1, \dots, T_t$ are independent indeterminates, ι is a closed immersion, and π is the natural map. By [I, p. 252, Prop. 4.4(ii) and p. 174, Prop. 2.2.9(b)], the sheaf $\iota_*\mathcal{O}_X$ is $\mathbf{D}(Y_n)$-isomorphic to a bounded quasi-coherent complex G of direct summands of finite-rank free $\mathcal{O}_{Y_n}$-modules, vanishing in all degrees $< -d-t$. Hence $\mathbf{R}f_*\mathcal{O}_X \cong \pi_*\iota_*\mathcal{O}_X \cong \pi_*G$ is $(-d-t)$_locally projective.

Since $\mathbf{R}f_*$ commutes with direct sums in $\mathbf{D}_{\mathsf{qc}}$ (because $\mathbf{R}f_*$ has a right adjoint, or more directly, by [Lp, 3.9.3.3]), it follows that for any free $\mathcal{O}_X$-module E, $\mathbf{R}f_*E$ is $(-d-t)$_locally projective. Finally, to show that for any a_locally projective E, $\mathbf{R}f_*E$ is $(a-d-t)$_locally projective, one reduces easily to where E is a bounded free complex, and then argues by induction on the number of degrees in which E is nonvanishing, using the following observation:

$(*)$: *In a $\mathbf{D}(X)$-triangle $N[-1] \xrightarrow{\delta} L \to M \xrightarrow{\rho} N$, if N is a_locally projective then M is a_locally projective iff so is L.*

(To see this, one may suppose that X is affine, say $X = \mathrm{Spec}(A)$. If N and L are a_locally projective, then one may assume they are sheafifications of bounded projective A-complexes vanishing in all degrees $< a$, so that by the dual versions of [Lp, (2.3.4) and (2.3.8)(v)], δ comes from a $\mathbf{K}(X)$-morphism $\delta_0\colon N[-1] \to L$; and M is isomorphic to the mapping cone of δ_0, an a_projective complex. Similarly, if N and M are sheafifications of bounded projective A-complexes vanishing in all degrees $< a$, then ρ comes from a $\mathbf{K}(X)$-morphism $\rho_0\colon M \to N$, and since $L[1]$ is isomorphic to the mapping cone of ρ_0, therefore L is a_locally projective.)

Suppose next that $X = X_1 \cup X_2$ with X_1 and X_2 quasi-compact open subsets for which there exists a $t > 0$ such that Lemma 3.3 holds with this t for all three of X_1, X_2, and $X_1 \cap X_2$. As in the proof of Lemma 3.2, there is a $\mathbf{D}(Y)$-triangle

$$\mathbf{R}f_*E \longrightarrow \mathbf{R}(fi)_*(E|_{X_1}) \oplus \mathbf{R}(fj)_*(E|_{X_2}) \longrightarrow \mathbf{R}(fk)_*(E|_{X_1\cap X_2}) \longrightarrow \mathbf{R}f_*E[1]$$

in which the two vertices other than $\mathbf{R}f_*E$ are $(a-d-t)$_projective, whence, by $(*)$, so is $\mathbf{R}f_*E$.

As before, this completes the proof of Lemma 3.3, and so of Theorem 3.1.

4. Approximation by perfect complexes.

Terminology remains as in §2.

The main results in this section are the following two theorems.

Theorem 4.1. *For any scheme X, there exists a positive integer $B = B(X)$ such that for any $E \in \mathbf{D}_{\mathsf{qc}}(X)$ and integer m, if E is $(m-B)$-pseudo-coherent then there exists in $\mathbf{D}_{\mathsf{qc}}(X)$ an m-isomorphism $P \to E$ with P perfect.*

Theorem 4.2. *Let X be a scheme. Then $\mathbf{D}_{\mathsf{qc}}(X)$ has a perfect generator, i.e., there is a perfect $\mathcal{O}_X$-complex S such that for each $E \neq 0$ in $\mathbf{D}_{\mathsf{qc}}(X)$ there is an $n \in \mathbb{Z}$ and a nonzero $\mathbf{D}_{\mathsf{qc}}(X)$-morphism $S[n] \to E$.*

Moreover, for each such S there is an integer $A = A(S)$ such that for all $E \in \mathbf{D}_{\mathsf{qc}}(X)$ and $j \in \mathbb{Z}$ with $H^j(E) \neq 0$,

$$\operatorname{Hom}\big(S[n], E\big) \neq 0 \quad \text{for some } \ n \leq A - j.$$

Theorem 4.1 may be compared to [I, p. 173, 2.2.8(b)]). The first statement in Theorem 4.2 comes from [BB, p. 9, Thm. 3.1.1].

Proofs are given in section 5 below.

Corollary 4.3.1. *If a map f is either perfect or quasi-perfect, then the functor $f^\times$ is bounded.*

Proof. As mentioned in the Introduction, $f^\times$ commutes with translation of complexes, and $f^\times$ is bounded below. So to show that $f^\times$ is bounded, it is enough to find a j_0 such that for every $m \in \mathbb{Z}$ and $F \in \mathbf{D}_{\mathsf{qc}}(Y)$ with $H^i(F) = 0$ for all $i > m$, it holds that $H^j f^\times F = 0$ for all $j \geq m + j_0$.

Suppose $H^j(f^\times F) \neq 0$. With S and A as in Theorem 4.2, there exists $k \leq A$ and a nonzero $\mathbf{D}(X)$-morphism $S \to f^\times F[j-k]$, the latter corresponding under adjunction to a *nonzero* morphism $\lambda\colon \mathbf{R}f_*S \to F[j-k]$.

For some a, $\mathbf{R}f_*S$ is a_locally projective—when f is perfect, that results from Lemma 3.3, and when f is quasi-perfect, it's because $\mathbf{R}f_*S$ is perfect. It follows from Lemma 3.2 that there is an integer $s = s(Y)$ such that λ cannot exist if $j \geq m + A - a + s$. With $j_0 := A - a + s$, we must have then that $H^j(f^\times F) = 0$ for all $j \geq m + j_0$; and so $f^\times$ is indeed bounded. □

Corollary 4.3.2. *For a map $f\colon X \to Y$, the following are equivalent.*

(i) *f is quasi-proper.*

(ii) *For any perfect $\mathcal{O}_X$-complex P, $\mathbf{R}f_*P$ is pseudo-coherent.*

(iii) *If S is as in Theorem 4.2, then $\mathbf{R}f_*S$ is pseudo-coherent.*

Proof. (i) ⇒ (ii) ⇒ (iii). The first implication is clear (since perfect complexes are pseudo-coherent); and the second is trivial.

(iii) ⇒ (ii). Let R be the smallest triangulated subcategory of $\mathbf{D}_{\mathsf{qc}}(X)$ containing S, and let $\widehat{R}$ be the full subcategory of $\mathbf{D}_{\mathsf{qc}}(X)$ whose objects are all the direct summands of objects in R. The subcategory $\widehat{R} \subset \mathbf{D}_{\mathsf{qc}}(X)$ is triangulated, and closed under formation of direct summands [N2, p. 99, 2.1.39].

The full subcategory R^c of R whose objects are the compact ones in R is triangulated, whence every object in R—and in $\widehat{R}$—is compact. Consequently, [N1, p. 222, Lemma 3.2] shows that the smallest full subcategory of $\mathbf{D}_{\mathsf{qc}}(X)$ which contains $\widehat{R}$ and is closed with respect to coproducts is $\mathbf{D}_{\mathsf{qc}}(X)$ itself.

Hence, by [N1, p. 214, Thm. 2.1.3], every perfect complex lies in $\widehat{R}$. (Alternatively, see [N2, p. 285, Prop. 8.4.1 and p. 140, Lemma 4.4.5].)

Since the pseudo-coherent complexes in $\mathbf{D}_{\mathsf{qc}}(Y)$ are the objects of a triangulated subcategory closed under formation of direct summands [I, p. 99, b) and p. 105, 2.12], therefore the complexes $Q \in \mathbf{D}_{\mathsf{qc}}(X)$ such that $\mathbf{R}f_*Q$ is pseudo-coherent are the objects of a triangulated subcategory closed under formation of direct summands. Thus if S is such a Q then every complex in $\widehat{R}$—and so every perfect complex—is such a Q.

(ii) ⇒ (i). Let E be a pseudo-coherent $\mathcal{O}_X$-complex, let $m \in \mathbb{Z}$, and let

$$P \xrightarrow{\ \alpha\ } E \longrightarrow Q \longrightarrow P[1]$$

be a triangle with α an m-isomorphism as in Theorem 4.1. Thus $H^k(Q) = 0$ for all $k \geq m$. As $\mathbf{R}f_*$ is bounded above [Lp, (3.9.2)], there is an integer t depending only on f such that $H^k(\mathbf{R}f_*Q) = 0$ for all $k \geq m+t$, that is, $\mathbf{R}f_*\alpha$ is an $(m+t)$-quasi-isomorphism. So if $\mathbf{R}f_*P$ is pseudo-coherent then $\mathbf{R}f_*E$ is $(m+t)$-pseudo-coherent; and since m is arbitrary, therefore $\mathbf{R}f_*E$ is pseudo-coherent. □

From 4.3.2(ii) we get:

Corollary 4.3.3. *Every quasi-perfect map is quasi-proper.*

Next, we deduce "stability" of quasi-properness.

Proposition 4.4. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ u\ ]{} & Y \end{array}$$

be a tor-independent square. If f is quasi-proper then so is g.

Proof. Since pseudo-coherence is a local property, it suffices to prove the Proposition when Y' is affine and $u(Y')$ is contained in an affine subset of Y. So we can assume that $u = u'u''$ where u' is an open immersion and u'' is an affine map. It follows that it suffices to prove the Proposition (a) when u—hence v—is an open immersion and (b) when u—hence v—is an affine map (see [GD, p. 358, (9.1.16)(iii), (9.1.17)]).

In either of these two cases, it holds that

$(*)$ *if S is as in Theorem 4.2 then $\mathbf{L}v^*S$ is a generator of $\mathbf{D}_{\mathsf{qc}}(X')$.*

Indeed, in case v is an open immersion and $0 \neq E \in \mathbf{D}_{\mathsf{qc}}(X')$ then $0 \neq \mathbf{R}v_*E \in \mathbf{D}_{\mathsf{qc}}(X)$ (since $E \cong v^*\mathbf{R}v_*E$); and the same holds in case v is affine, by [Lp, (3.10.2.2]. So in either case, for some n,

$$0 \neq \mathrm{Hom}_{\mathbf{D}_{\mathsf{qc}}(X)}(S[n], \mathbf{R}v_*E) \cong \mathrm{Hom}_{\mathbf{D}_{\mathsf{qc}}(X')}(\mathbf{L}v^*S[n], E),$$

proving $(*)$.

It is easy to see that the complex $\mathbf{L}v^*S$ is perfect. So by Corollary 4.3.2, to prove the Proposition for u as in $(*)$ it suffices to show that $\mathbf{R}g_*\mathbf{L}v^*S$ is pseudo-coherent. But by [Lp, (3.10.3)], $\mathbf{R}g_*\mathbf{L}v^*S \cong \mathbf{L}u^*\mathbf{R}f_*S$; and since $\mathbf{R}f_*S$ is pseudo-coherent, therefore, by [I, p. 111, 2.16.1], so is $\mathbf{L}u^*\mathbf{R}f_*S$.

5. Proofs of Theorems 4.1 and 4.2

Heavy use will be made of the following technical notion.

Definition 5.1. Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{S} \subset \mathcal{T}$ be a class of objects. Let $m \le n$ be integers. The full subcategory $\mathcal{S}[m,n] \subset \mathcal{T}$ is the smallest among (= intersection of) all full subcategories $\mathscr{S} \subset \mathcal{T}$ such that:

(i) 0 is contained in $\mathscr{S}$.
(ii) If $E \in \mathcal{S}$, then $E[-\ell] \in \mathscr{S}$ for all integers ℓ in the interval $[m,n]$.
(iii) For any $\mathcal{T}$-triangle

$$E \longrightarrow F \longrightarrow G \longrightarrow E[1],$$

if E and G are in $\mathscr{S}$ then so is F.

Remark 5.2. One checks that $\mathcal{S}[m,n] = \big(\bigcup_{\ell=m}^{n} \mathcal{S}[-\ell]\big)[0,0]$.

Remark 5.3. Defn. 5.1 expands to allow $m = -\infty$ or $n = \infty$. For example, $\mathcal{S}[m,\infty) := \bigcup_{n=m}^{\infty} \mathcal{S}[m,n]$. Furthermore, $\mathcal{S}(\infty,\infty) := \bigcup_{m\le n} \mathcal{S}[m,n]$, being closed under translation (see 5.4(i)), is the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{S}$ [N2, p. 60, Defn. 1.5.1].

Remark 5.4. The following are easy observations.

(i) *If* $E \in \mathcal{S}[m,n]$ *and* $j \in \mathbb{Z}$ *then* $E[-j] \in \mathcal{S}[m+j, n+j]$.
Indeed, (i), (ii) and (iii) in 5.1 hold for the full subcategory $\mathscr{S} \subset \mathcal{T}$ whose objects are those $E \in \mathcal{S}[m,n]$ such that $E[-j] \in \mathcal{S}[m+j,n+j]$. One deduces that, with $\mathcal{S}[m,n]_\text{o}$ the class of objects in $\mathcal{S}[m,n]$,

$$\big(\mathcal{S}[m,n]_\text{o}\big)[m',n'] = \mathcal{S}[m+m', n+n'].$$

(ii) *If every object of* $\mathcal{S}$ *is compact, then so is every object of* $\mathcal{S}[m,n]$.
Indeed, (i), (ii) and (iii) in 5.1 hold for the full subcategory $\mathscr{S} \subset \mathcal{T}$ whose objects are those $E \in \mathcal{S}[m,n]$ which are compact.

(iii) *Let* $\mathcal{A}$ *be an abelian category, and* $H : \mathcal{T} \longrightarrow \mathcal{A}$ *a cohomological functor, see* [N2, p. 32, 1.1.9]. *If for every object* $F \in \mathcal{S}$ *we have* $H(F[-i]) = 0$ *for all* i *in some interval* $[a,b]$, *then for all* $E \in \mathcal{S}[m,n]$,

$$H(E[-j]) = 0 \text{ for all } j \in [a-m, b-n]. \tag{5.4.1}$$

Indeed, (i), (ii) and (iii) in 5.1 hold for the full subcategory $\mathscr{S} \subset \mathcal{T}$ whose objects are those $E \in \mathcal{S}[m,n]$ which satisfy (5.4.1).

(iv) *Let* $\phi\colon \mathcal{T} \to \mathcal{T}'$ *be a triangle-preserving additive functor* [Lp, §1.5]. *Then*

$$\phi\big(\mathcal{S}[m,n]\big) \subset \big\{\phi\mathcal{S}\big\}[m,n].$$

Indeed, (i), (ii) and (iii) in 5.1 hold for the full subcategory $\mathcal{S} \subset \mathcal{T}$ whose objects are those $E \in \mathcal{S}[m,n]$ such that $\phi E \in \big\{\phi\mathcal{S}\big\}[m,n]$.

(v) *Let* $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ *be morphisms inside* $\mathcal{T}$*-triangles*

$$E \longrightarrow A \xrightarrow{\ \alpha\ } B \longrightarrow E[1],$$

$$F \longrightarrow A \xrightarrow{\ \beta\alpha\ } C \longrightarrow F[1],$$

$$G \longrightarrow B \xrightarrow{\ \beta\ } C \longrightarrow G[1].$$

If E *and* G *are in* $\mathcal{S}[m,n]$ *then so is* F.

Indeed, the octahedral axiom [N2, p. 60, 1.4.7] produces a triangle

$$E \longrightarrow F \longrightarrow G \longrightarrow E[1].$$

Example 5.5. Remark 5.4(iii) will be used thus. Let G be an object of $\mathcal{T}$, and H the cohomological functor $H(-) := \mathrm{Hom}(-,G)$, see [N2, p. 33, 1.1.11]. Then for $a = m$ and $b = n$ the assertion becomes:

If for every object $F \in \mathcal{S}$ *we have* $\mathrm{Hom}(F[-i],G) = 0$ *for all* $i \in [m,n]$, *then* $\mathrm{Hom}(E,G) = 0$ *for all* $E \in \mathcal{S}[m,n]$.

A key role in the proofs will be played by *Koszul complexes.*

Example 5.6. Let R be a commutative ring, $(f_1, f_2, \dots, f_r)$ a sequence in R, and $(n_1, n_2, \dots, n_r)$ a sequence of positive integers. The associated Koszul complex (see, e.g., [EGA, III, (1.1.1)]) is

$$K_\bullet(f_1^{n_1}, \dots, f_r^{n_r}) := \otimes_{i=1}^r K_\bullet(f_i^{n_i}),$$

where $K_\bullet(f_i^{n_i})$ is $R \xrightarrow{f_i^n} R$ in degrees -1 and 0, and (0) elsewhere.

For $r = 0$, set $K_\bullet(\phi) := R$. For all $r \geq 0$, $K_\bullet(f_1^{n_1}, \dots, f_r^{n_r})$ is a complex with perfect amplitude in $[-r,0]$, and homology killed by each $f_i^{n_i}$.

For any complex E, and $f \in R$, $K_\bullet(f) \otimes E$ is the mapping cone of the endomorphism "multiplication by f" of E. Thus for $1 \leq i < r$, $K_\bullet(f_i^{n_i}, \dots, f_r^{n_r})$ is the mapping cone of the endomorphism "multiplication by $f_i^{n_i}$" of the complex $K_\bullet(f_{i+1}^{n_{i+1}}, f_{i+2}^{n_{t+2}}, \dots, f_r^{n_r})$. It follows that

$$K_\bullet(f_1^{n_1}, f_2^{n_2}, \dots, f_r^{n_r}) \in \{K_\bullet(f_1, f_2, \dots, f_r)\}\,[0,0].$$

This is shown by a straightforward induction, based on application of 5.4(v) to the following three natural triangles (where $\widehat{\ }$ signifies "omit,"):

$$K_\bullet(f_1^{n_1}, .., f_i^{n_i}, f_{i+1}, .., f_r) \longrightarrow$$
$$K_\bullet(f_1^{n_1}, .., \widehat{f_i^{n_i}}, f_{i+1}, .., f_r)[1] \xrightarrow{f_i^{n_i}} K_\bullet(f_1^{n_1}, .., \widehat{f_i^{n_i}}, f_{i+1}, .., f_r)[1] \xrightarrow{+}$$

$$\begin{aligned}&K_\bullet(f_1^{n_1},..,f_i^{n_i+1},f_{i+1},..,f_r)\longrightarrow\\ &\quad K_\bullet(f_1^{n_1},..,\widehat{f_i^{n_i+1}},f_{i+1},..,f_r)[1]\xrightarrow{f_i^{n_i+1}} K_\bullet(f_1^{n_1},..,\widehat{f_i^{n_i+1}},f_{i+1},..,f_r)[1]\xrightarrow{+}\end{aligned}$$

$$\begin{aligned}&K_\bullet(f_1^{n_1},..,f_i,f_{i+1},..,f_r)\longrightarrow\\ &\qquad K_\bullet(f_1^{n_1},..,\widehat{f_i},f_{i+1},..,f_r)[1]\xrightarrow{f_i} K_\bullet(f_1^{n_1},..,\widehat{f_i},f_{i+1},..,f_r)[1]\xrightarrow{+}\end{aligned}$$

The proofs of Theorems 4.1 and 4.2 will involve induction on the number of affine open subschemes needed to cover X. One needs to begin with some results on affine schemes.

In the situation of Example 5.6, denote the sequence $(f_1^n,\dots,f_r^n)$ $(n>0)$ by $\mathbf{f}^n$, omitting the superscript "n" when $n=1$. Let $C_\bullet(\mathbf{f}^n)$ be the cokernel of that map of complexes $R[-1]\to K_\bullet(\mathbf{f}^n)[-1]$ which is the identity map of R in degree 1. The complex $C_\bullet(\mathbf{f}^n)$ has perfect amplitude in $[1-r,0]$; and there is a natural homotopy triangle

$$C_\bullet(\mathbf{f}^n)\longrightarrow R\longrightarrow K_\bullet(\mathbf{f}^n)\longrightarrow C_\bullet(\mathbf{f}^n)[1]. \tag{5.6.1}$$

There is a map of complexes $K_\bullet(f^{n+m})\to K_\bullet(f^n)$ $(f\in R;\ m,n>0)$ depicted by

$$\begin{array}{ccc} R & = & R\\ {\scriptstyle f^{n+m}}\uparrow & & \uparrow{\scriptstyle f^n}\\ R & \xrightarrow[f^m]{} & R\end{array}$$

Tensoring such maps gives a map $K_\bullet(\mathbf{f}^{n+m})\to K_\bullet(\mathbf{f}^n)$, and hence a map $C_\bullet(\mathbf{f}^{n+m})\to C_\bullet(\mathbf{f}^n)$. For any R-complex E, we have then the *Čech complex*

$$\check{C}^\bullet(\mathbf{f},E):=\varinjlim_n \operatorname{Hom}^\bullet_R(C_\bullet(\mathbf{f}^n),E).$$

Let U be the complement of the closed subscheme $\operatorname{Spec}(R/\mathbf{f}R)\subset\operatorname{Spec}(R)$, with inclusion $\iota\colon U\hookrightarrow X$. From [EGA, III, §1.3] it follows readily that, with $E^\sim$ the quasi-coherent complex corresponding to E, $\mathbf{R}\iota_*\iota^*E^\sim$ *is naturally* $\mathbf{D}$*-isomorphic to the sheafified Čech complex* $\check{\mathcal{C}}^\bullet(\mathbf{f},E):=\check{C}^\bullet(\mathbf{f},E)^\sim$.

In particular, if the homology of E is $\mathbf{f}R$-torsion (i.e., for all $i\in\mathbb{Z}$, each element of $H^i(E)$ is annihilated by a power of $\mathbf{f}R$)—or equivalently, if $\iota^*E^\sim$ is exact—then $\check{C}^\bullet(\mathbf{f},E)$ is exact; and since the complex $C_\bullet(\mathbf{f}^n)$ is bounded and projective, therefore

$$H^0\operatorname{Hom}^\bullet_R(C_\bullet(\mathbf{f}^n),E)\cong H^0\mathbf{R}\operatorname{Hom}^\bullet_R(C_\bullet(\mathbf{f}^n),E)\cong\operatorname{Hom}_{\mathbf{D}(R)}(C_\bullet(\mathbf{f}^n),E),$$

whence

$$\varinjlim_n\operatorname{Hom}_{\mathbf{D}(R)}(C_\bullet(\mathbf{f}^n),E)\cong H^0\check{C}^\bullet(\mathbf{f},E)=0.$$

Consequently, the commutative diagrams of the following form, with exact rows coming from (5.6.1), and columns from the maps described above:

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbf{D}(R)}(K_\bullet(\mathbf{f}),E) & \longrightarrow & \mathrm{Hom}_{\mathbf{D}(R)}(R,E) & \longrightarrow & \mathrm{Hom}_{\mathbf{D}(R)}(C_\bullet(\mathbf{f}),E) \\ \downarrow & & \| & & \downarrow \\ \mathrm{Hom}_{\mathbf{D}(R)}(K_\bullet(\mathbf{f}^n),E) & \longrightarrow & \mathrm{Hom}_{\mathbf{D}(R)}(R,E) & \longrightarrow & \mathrm{Hom}_{\mathbf{D}(R)}(C_\bullet(\mathbf{f}^n),E) \end{array}$$

show that *for any* $\lambda \in \mathrm{Hom}_{\mathbf{D}(R)}(R,E)$ *there is an* $n>0$ *such that* λ *factors through a* $\mathbf{D}(R)$*-morphism* $K_\bullet(\mathbf{f}^n)\to E$.

If P is a bounded complex of finitely generated projective R-modules, then the homology of $\mathrm{Hom}^\bullet_R(P,E)$ is still $\mathbf{f}R$-torsion (as one sees, e.g., by induction on the number of nonvanishing components of P); and replacing E in what precedes by $\mathrm{Hom}^\bullet_R(P,E)$, one obtains, via Hom-$\otimes$ adjunction, that *for any* $\lambda\in\mathrm{Hom}_{\mathbf{D}(R)}(P,E)$ *there is an* $n>0$ *such that* λ *factors through a* $\mathbf{D}(R)$*-morphism* $K_\bullet(\mathbf{f}^n)\otimes P\to E$.

Lemma 5.7. *Let* E *be an* R*-complex such that* $H^j(E)$ *is* $\mathbf{f}R$*-torsion for all* $j\ge -r$, P *an* R*-complex with perfect amplitude in* $[0,b]$ *for some* $b\ge 0$, *and* $\lambda\in\mathrm{Hom}_{\mathbf{D}(R)}(P,E)$. *Then there is an integer* $n>0$ *and a homomorphism of* R*-complexes* $\lambda_n\colon K_\bullet(\mathbf{f}^n)\otimes P\to E$ *such that for all* $j\ge -r$, *the homology map* $H^j(\lambda)\colon H^j(P)\to H^j(E)$ *factors as*

$$H^j(P)=H^j(R\otimes P)\xrightarrow{\text{natural}} H^j\big(K_\bullet(\mathbf{f}^n)\otimes P\big)\xrightarrow{H^j(\lambda_n)} H^j(E).$$

Proof. We may assume that P is a complex of finitely generated projective R-modules, vanishing in all degrees outside $[0,b]$, see [I, p. 175, b)]. Let $\tau_{\ge -r}E$ be the usual truncation, and $\pi\colon E\to\tau_{\ge -r}E$ the natural map, which induces homology isomorphisms in all degrees $\ge -r$ (see, e.g., [Lp, §1.10]). By the preceding remarks, $\pi\lambda$ factors in $\mathbf{D}(R)$ as

$$P=R\otimes P\xrightarrow{\text{natural}} K_\bullet(\mathbf{f}^n)\otimes P\xrightarrow{\bar\lambda_n}\tau_{\ge -r}E.$$

Since $K_\bullet(\mathbf{f}^n)\otimes P$ is bounded and projective, we may assume that $\bar\lambda_n$ is a map of R-complexes. Then the R-homomorphism

$$(\bar\lambda_n)^{-r}\colon \big(K_\bullet(\mathbf{f}^n)\otimes P\big)^{-r}=P^0\to(\tau_{\ge -r}E)^{-r}=\mathrm{coker}(E^{-r-1}\to E^{-r})$$

lifts to a map $P^0\to E^{-r}$, giving a map λ_n with the desired properties. □

Corollary 5.7.1. *Set* $I:=\mathbf{f}R=(f_1,f_2,\dots,f_r)R$. *Let* $m\in\mathbb{Z}$ *and let* E *be an* R*-complex such that* $H^i(E)$ *is* I*-torsion for all* $i\ge m-r$.

(i) *If* E *is* m*-pseudocoherent, and* $p\ge m$ *is such that* $H^i(E)=0$ *for all* $i>p$, *then there exists in the homotopy category of* R*-complexes an* m*-quasi-isomorphism* $P\to E$ *with* $P\in\{K_\bullet(\mathbf{f})\}[m,p]$.

(ii) *For any* $i\ge m$ *with* $H^i(E)\ne 0$, *there is a nonzero map* $K_\bullet(\mathbf{f})[-i]\to E$.

Proof. (i) By [I, p. 103, 2.10(b)], $H^p(E)$ is a finitely generated R-module. So there is an $\ell > 0$ and a surjective homomorphism $R^\ell \twoheadrightarrow H^p(E)$, which lifts to $R^\ell \to \ker(E^p \to E^{p+1})$, and thus there is a homomorphism $R^\ell \to E[p]$, or equivalently, $\lambda\colon R^\ell[m-p] \to E[m]$, giving rise, by Lemma 5.7, to an R-homomorphism

$$\lambda_n[-m]\colon P_1 := \big(K_\bullet(\mathbf{f}^n) \otimes R^\ell[-p]\big) \to E$$

such that $H^p\big(\lambda_n[-m]\big)$ is surjective. By Example 5.6 and Remark 5.4(i), we have $K_\bullet(\mathbf{f}^n)[-p] \in \{K_\bullet(\mathbf{f})\}[p,p]$. So we get a homotopy triangle

$$P_1 \longrightarrow E \xrightarrow{\ \alpha\ } Q_1 \longrightarrow P_1[1]$$

with $P_1 \in \{K_\bullet(\mathbf{f})\}[p,p]$ and $H^i(Q_1) = 0$ for all $i \ge p$, giving (i) when $p = m$.

In any case, Q_1 is m-pseudocoherent [I, p.100, 2.6]; and since all the homology of P_1 is I-torsion, the exact homology sequence of the preceding triangle shows that $H^i(Q_1)$ is I-torsion for all $i \ge m - r$. If $m < p$ then, using induction on $p - m$, one may assume that there is a homotopy triangle

$$P_2 \longrightarrow Q_1 \xrightarrow{\ \beta\ } Q \longrightarrow P_2[1]$$

with $P_2 \in \{K_\bullet(\mathbf{f})\}[m, p-1] \subset \{K_\bullet(\mathbf{f})\}[m,p]$ and $H^i(Q) = 0$ for all $i \ge m$. There exists then a homotopy triangle

$$P \longrightarrow E \xrightarrow{\ \beta\alpha\ } Q \longrightarrow P[1]$$

which, by Remark 5.4(v), is as desired.

(ii) There is, by assumption, a nonzero map $R \to H^i(E)$, which lifts to a map $R \to \ker(E^i \to E^{i+1})$; and so there is a nonzero map $\lambda\colon R \to E[i]$ with $H^0(\lambda) \ne 0$. If $j \ge -r$ then $j + i \ge i - r \ge m - r$, so $H^j(E[i]) = H^{j+i}(E)$ is I-torsion, whence by Lemma 5.7, there is for some $n > 0$ a nonzero map $K_\bullet(\mathbf{f}^n) \to E[i]$. By Example 5.6, $K_\bullet(\mathbf{f}^n) \in K_\bullet(\mathbf{f})[0,0]$; so by Example 5.5, there is a nonzero map $K_\bullet(\mathbf{f}) \to E[i]$, proving (ii). □

For dealing with the nonaffine situation, we need to set up some notation.

Notation 5.8. A scheme X can be covered by finitely many open affine subsets, say $X = \bigcup_{k=1}^t U_k$, with $U_k = \mathrm{Spec}(R_k)$. For $1 \le k \le t$, set

(i) $V_k := \bigcup_{i=k}^t U_i$.
(ii) $Y_k := X - V_{k+1}$ ($:= X$ when $k = t$).

So we have a filtration by closed subschemes $Y_1 \subset Y_2 \subset \cdots \subset Y_t = X$.

Both U_k and V_{k+1} are quasi-compact open subsets of the (quasi-separated) scheme X, whence so is $U_k \cap V_{k+1}$. So there is a sequence

$$\mathbf{f}_k = \{f_{k1}, f_{k2}, \cdots, f_{kr_k}\}$$

in R_k such that

$$U_k \cap V_{k+1} \;=\; \bigcup_{i=1}^{r_k} \mathrm{Spec}(R_k[1/f_{ki}])\ .$$

Set

(iii) $I_k := \mathbf{f}_k R_k$, (so that $U_k \cap V_{k+1}$ is the complement of the closed subscheme $\operatorname{Spec}(R_k/I_k) \subset U_k$).

(1) (iv) $C_k := \big(K_\bullet(\mathbf{f}_k) \oplus K_\bullet(\mathbf{f}_k)[1]\big)^\sim = \big(K_\bullet(0, f_{k1}, f_{k2}, \cdots, f_{kr_k})\big)^\sim$
with $K_\bullet(*)$ the Koszul complex over R_k associated to the sequence $(*)$, and $(-)^\sim$ the sheafification functor from R_k-modules to quasi-coherent $\mathcal{O}_{U_k}$-modules—so that C_k is a perfect $\mathcal{O}_{U_k}$-complex.

(The reason for introducing this $\oplus$ will emerge shortly.)

We have the cartesian diagram of (open) inclusion maps

$$\begin{array}{ccc} U_k \cap V_{k+1} & \xrightarrow{\ \nu\ } & V_{k+1} \\ {\scriptstyle\lambda}\big\downarrow & & \big\downarrow{\scriptstyle\xi} \\ U_k & \xrightarrow[\ \mu\ ]{} & V_k = U_k \cup V_{k+1} \end{array}$$

The restriction $\lambda^* C_k$ is homotopically trivial, whence, in $\mathbf{D}(V_{k+1})$,

$$\xi^* \mathbf{R}\mu_* C_k \cong \mathbf{R}\nu_* \lambda^* C_k = 0.$$

Thus, the restrictions of $\mathbf{R}\mu_* C_k$ to both V_{k+1} and U_k are perfect, and so $\mathbf{R}\mu_* C_k$ is itself perfect.

For any $\mathcal{O}_{V_k}$-complex G, the obvious triangle

$$G \xrightarrow{\ 0\ } G \longrightarrow G \oplus G[1] \longrightarrow G[1]$$

shows that the complex $G \oplus G[1]$ vanishes in the Grothendieck group $\mathcal{K}_0(V_k)$. Taking $G := \mathbf{R}\mu_*\big(K_\bullet(\mathbf{f}_k)\big)^\sim$, we deduce then from Thomason's localization theorem [TT, p. 338, 5.2.2(a)] that the perfect $\mathcal{O}_{V_k}$-complex $\mathbf{R}\mu_* C_k$ is $\mathbf{D}(V_k)$-isomorphic to the restriction of a perfect $\mathcal{O}_X$-complex.

(v) Let $S_k \in \mathbf{D}_{\mathsf{qc}}(X)$ be a perfect $\mathcal{O}_X$-complex whose restriction to V_k is $\mathbf{D}(V_k)$-isomorphic to $\mathbf{R}\mu_* C_k$.

(vi) Let $\mathcal{S}_k$ be the finite set $\{S_1, S_2, \dots, S_k\}$.

According to Lemma 3.2, there is for each k an integer $N_k > 0$ such that, if $Q \in \mathbf{D}_{\mathsf{qc}}(X)$ satisfies $H^\ell(Q) = 0$ for all $\ell \ge -N_k$ then $\operatorname{Hom}_{\mathbf{D}(X)}(S_k, Q) = 0$. After enlarging N_k if necessary, we have also that $\operatorname{Hom}_{\mathbf{D}(X)}(Q, S_k) = 0$.

Set

(vii) $N := \max\{\, N_1, N_2, \dots, N_t, r_1, r_2, \dots, r_t \,\} + 1.$

Next comes the key statement.

Proposition 5.9. *With the preceding notation, let $m, k \in \mathbb{Z}$, $1 \le k \le t$, let $E \in \mathbf{D}_{\mathsf{qc}}(X)$ be such that $H^j(E)$ is supported in Y_k for all $j \ge m - kN$, and set*

$$a_k = \binom{k+1}{2} N \qquad (1 \le k \le t).$$

(i) *If E is $(m-(k-1)N)$-pseudo-coherent then there is an m-isomorphism $P \to E$ with $P \in \mathcal{S}_k[m-a_k, \infty)$ (so that P is perfect, see* 5.4(ii)).

(ii) *If $H^\ell(E) \neq 0$ for some $\ell \geq m$, then for some $i \geq m - a_k$ and some $j \in [1,k]$, there is a nonzero map $S_j[-i] \to E$.*

Before proving this, let us see how to derive Theorems 4.1 and 4.2.

Since $Y_t = X$, Theorem 4.1, with $B := (t-1)N$, is contained in 5.9(i).

Next, 5.9(ii) with $k=t$ shows that if $H^\ell(E) \neq 0$, then there exist integers $i \geq \ell - a_t$ and $j \in [1,t]$, and a non-zero map $S_j[-i] \to E$. This gives Theorem 4.2 for the specific choices

$$S = S_1 \oplus S_2 \oplus \cdots \oplus S_t, \qquad A(S) := a_t = \binom{t+1}{2} N\,.$$

The rest of Theorem 4.2 results from the following general fact, applied to $\mathcal{H} = \{\, E \in \mathbf{D}_{\mathsf{qc}}(X) \mid H^\ell(E) \neq 0 \,\}$, $\mathcal{T} = \mathbf{D}_{\mathsf{qc}}(X)$ and $A = A(S) - \ell$.

Proposition 5.10. *Let $\mathcal{T}$ be a triangulated category with coproducts. Let $\mathcal{H}$ be a collection of objects of $\mathcal{T}$. Suppose there exists a compact generator $S \in \mathcal{T}$ and an integer A such that*

$$E \in \mathcal{H} \implies \operatorname{Hom}\big(S[n], E\big) \neq 0 \text{ for some } n \leq A.$$

Then every compact generator has a similar property: for each compact generator $S' \in \mathcal{T}$ there is an integer A' such that

$$E \in \mathcal{H} \implies \operatorname{Hom}\big(S'[n], E\big) \neq 0 \text{ for some } n \leq A'.$$

Proof. Let $\widehat{R}$ be the full subcategory of $\mathcal{T}$ whose objects are all the direct summands of objects in $\{S'\}(-\infty,\infty) = \bigcup_{M\geq 0}\{S'\}[-M,M]$ (see Remark 5.3). As in the proof of Corollary 4.3.2, (iii) ⇒ (ii), one sees that $S \in \widehat{R}$, i.e., there is an $S^* \in \widehat{R}$ and an $M \geq 0$ such that $S \oplus S^* \in \{S'\}[-M,M]$.

Now if $E \in \mathcal{H}$ then, since $\operatorname{Hom}\big(S[k],E\big) \neq 0$ for some $k \leq A$, and $S[k] \oplus S^*[k] \in \{S'\}[-M-k, M-k]$ (Remark 5.4(i)), therefore Example 5.5 gives $\operatorname{Hom}(S'[n],E) \neq 0$ for some n with $n \leq M + k \leq M + A$. □

It remains to prove Proposition 5.9, which we do now by induction on k.

For $k=1$, $a_1 = N$, so $H^j(E)$ is supported in $Y_1 = \operatorname{Spec}(R_1/I_1)$ for all $j \geq m - r_1 - 1$ $(\geq m-N)$. As usual, when considering the restriction $E|_{U_1}$ we may assume it to be a quasi-coherent complex, then relate facts about it to facts about the corresponding complex E of R_1-modules. For example, it holds that $H^i(\mathrm{E})$ is I_1-torsion for all $i \geq m - r_1 - 1$.

Thus, from Corollary 5.7.1(i), applied to $I_1 = (0, f_{11}, f_{22}, \cdots, f_{1r_1})R_1$, it follows via 5.8(iv)) that, if E is m-pseudo-coherent then there exists an m-isomorphism $P \to E|_{U_1}$ with $P \in \{C_1\}[m,\infty)$. Likewise (and more easily), Corollary 5.7.1(ii) gives that if $H^i(E) \neq 0$ for some $i \geq m$—whence, $H^i(E)$ being supported in $Y_1 \subset U_1$, $H^i(E|_{U_1}) \neq 0$—then there is a nonzero map

$$C_1[-i] = \big(K_\bullet(\mathbf{f}_1)[-i]\big)^\sim \oplus \big(K_\bullet(\mathbf{f}_1)[-i+1]\big)^\sim \to E|_{U_1}.$$

Let $\mu\colon U_1 \hookrightarrow X$ be the inclusion. Note that $Y_1 = U_1 \setminus V_2$. Since C_1 is exact outside Y_1, so is $P \in \{C_1\}[m,\infty)$ (argue as in Remark 5.4(i)–(iv)), as is $\mathbf{R}\mu_*P$; and by assumption, $H^i(E)$ vanishes outside Y_1 for all $i \ge m$. With all this in mind, we can extend the preceding statements from U_1 to $X = U_1 \cup V_2$, by applying the following Lemma to $U = U_1$, $V = V_2$, and $C = C_1$ or P.

Lemma 5.11. *Let U and V be open subsets of a scheme X, and let*

$$\begin{array}{ccc} U\cap V & \xrightarrow{\ \nu\ } & V \\ {\scriptstyle\lambda}\big\downarrow & & \big\downarrow{\scriptstyle\xi} \\ U & \xrightarrow[\ \mu\ ]{} & U\cup V \end{array}$$

be the natural diagram of inclusion maps. Let $C \in \mathbf{D}(U)$ satisfy $\lambda^*C = 0$. Let $E \in \mathbf{D}(U\cup V)$. Then:

(i) *Every $\mathbf{D}(U)$-morphism $C \to \mu^*E$ extends uniquely to a $\mathbf{D}(U\cup V)$-morphism $\mathbf{R}\mu_*C \to E$.*

(ii) *If C is perfect then so is $\mathbf{R}\mu_*C$.*

(iii) *If $\mathcal{S} \subset \mathbf{D}(U)$ and $m \le n \in \mathbb{Z}$ then $\mathbf{R}\mu_*\big(\mathcal{S}[m,n]\big) \subset \big\{\mathbf{R}\mu_*\mathcal{S}\big\}[m,n]$.*

Proof. (i) In view of the natural isomorphisms

$$\mathrm{Hom}_{\mathbf{D}(U)}(C,\mu^*E) \cong \mathrm{Hom}_{\mathbf{D}(U)}(\mu^*\mathbf{R}\mu_*C,\mu^*E) \cong \mathrm{Hom}_{\mathbf{D}(U\cup V)}(\mathbf{R}\mu_*C,\mathbf{R}\mu_*\mu^*E)$$

we need only show that the natural map is an isomorphism

$$\mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\mu_*C,E) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\mu_*C,\mathbf{R}\mu_*\mu^*E)$$

(to which we can apply the homology functor H^0). Thus for any triangle

$$G \longrightarrow E \xrightarrow{\text{natural}} \mathbf{R}\mu_*\mu^*E \longrightarrow G[1]$$

we'd like to see that $\mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\mu_*C,G) = 0$. But $\mu^*G = 0 = \lambda^*C$, so that

$$\mu^*\mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\mu_*C,G) \cong \mathbf{R}\mathrm{Hom}^\bullet(\mu^*\mathbf{R}\mu_*C,\mu^*G) = 0, \quad \text{and}$$
$$\xi^*\mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\mu_*C,G) \cong \mathbf{R}\mathrm{Hom}^\bullet(\xi^*\mathbf{R}\mu_*C,\xi^*G) \cong \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\nu_*\lambda^*C,\xi^*G) = 0,$$

whence the conclusion.

(ii) Since both $\xi^*\mathbf{R}\mu^*C \cong \mathbf{R}\nu_*\lambda^*C = 0$ and $\mu^*\mathbf{R}\mu_*C = C$ are perfect, therefore so is $\mathbf{R}\mu_*C$.

(iii) This is a special case of Remark 5.4(iv). □

Lemma 5.12. *For $k > 1$, suppose Proposition 5.9(i) holds with $k-1$ in place of k. Then for any $E \in \mathbf{D}_{\mathsf{qc}}(X)$ and $\mathbf{D}(U_k)$-morphism*

$$\psi\colon F \longrightarrow E|_{U_k} \qquad \big(F \in \{C_k\}[m,\infty)\big),$$

there exists a $\mathbf{D}(X)$-morphism

$$\tilde\psi\colon \widetilde F \to E \qquad \big(\widetilde F \in \mathcal{S}_k[m-N-a_{k-1}\,,\,\infty)\big)$$

whose restriction $\tilde\psi|_{U_k}$ is isomorphic to ψ.

Before proving this Lemma, let us see how it is used to establish the induction step in the proof of Proposition 5.9. With reference to that Proposition, we show, for $k > 1$:

(1) Assertion (i) for $k-1$ implies assertion (i) for k.

(2) Assertions (i) and (ii) for $k-1$, together, imply assertion (ii) for k.

To prove (1), let $E \in \mathbf{D}_{\mathsf{qc}}(X)$ be $(m-(k-1)N)$-pseudocoherent, with $H^j(E)$ supported in Y_k for all $j \ge m-kN$. Since $m-(k-1)N-r_k \ge m-kN$, therefore (after replacement of $K_\bullet(\mathbf{f}_k)^\sim$ by C_k, see above) Corollary 5.7.1 provides a $\mathbf{D}(U_k)$-triangle

$$P_k \longrightarrow E|_{U_k} \longrightarrow Q_k \longrightarrow P_k[1] \tag{5.12.1}$$

with $P_k \in \{C_k\}[m-(k-1)N,\,\infty)$ and $H^j(Q_k)=0$ for all $j \ge m-(k-1)N$. By Lemma 5.12, the map $P_k \longrightarrow E|_{U_k}$ is isomorphic to the restriction of a $\mathbf{D}(X)$-morphism $\psi'\colon P' \to E$, with $P' \in \mathcal{S}_k[m-(k-1)N-N-a_{k-1}\,,\,\infty)$, i.e., since

$$a_{k-1}+kN \quad=\quad \binom{k}{2}N+kN \quad=\quad \binom{k+1}{2}N \quad=\quad a_k\,,$$

with $P' \in \mathcal{S}_k[m-a_k,\infty)$. Any $\mathbf{D}_{\mathsf{qc}}(X)$-triangle

$$P' \xrightarrow{\ \psi'\ } E \xrightarrow{\ \alpha\ } Q' \longrightarrow P'[1]\,,$$

restricts on U_k to one isomorphic to (5.12.1). So when $j \ge m-(k-1)N$, then $H^j(Q')$ vanishes on U_k; furthermore, $H^j(E)$ is supported on Y_k, and since all the members of $\mathcal{S}_k$ are exact outside Y_k therefore so is P' (argue as in Remark 5.4(i)–(iv)); and thus $H^j(Q')$ is supported in $(Y_k \setminus U_k) = Y_{k-1}$.

Moreover, Q' is $(m-(k-2)N)$-pseudocoherent, since both P' and E are [I, p. 100, 2.6]. So now the inductive assumption produces a triangle

$$P'' \longrightarrow Q' \xrightarrow{\ \beta\ } Q \longrightarrow P''[1]$$

with $P'' \in \mathcal{S}_{k-1}[m-a_{k-1},\infty)$, and $H^j(Q)=0$ whenever $j \ge m$.

There is then a triangle

$$P \xrightarrow{\ \psi'\ } E \xrightarrow{\ \beta\alpha\ } Q \longrightarrow P[1]\,,$$

and the assertion 5.9(i), for the integer k, results from Remark 5.4(v).

As for (2), let E satisfy the hypotheses of 5.9(ii) for k. If $H^i(E|_{U_k})=0$ for all $i \ge m-(k-1)N$ then $H^j(E)$ is supported in Y_{k-1} for all $j \ge m-(k-1)N$, $H^\ell(E)$ is non-zero for some $\ell \ge m$, and $m-a_{k-1} \ge m-a_k$; so in this case assertion (ii) for k is already given by assertion (ii) for $k-1$.

If, on the other hand, $H^i(E|_{U_k}) \ne 0$ for some $i \ge m-(k-1)N$, then, since $m-(k-1)N-r_k \ge m-kN$, Corollary 5.7.1 (suitably modified) provides a nonzero map $C_k[-i] \to E|_{U_k}$. By Remark 5.4(i),

$$C_k[-i] \in \{C_k\}[i,\infty) \subset \{C_k\}[m-(k-1)N,\infty),$$

so by Lemma 5.12, there exists a nonzero $\mathbf{D}(X)$-morphism $\widetilde{F} \to E$ with

$$\widetilde{F} \in \mathcal{S}_k[m-(k-1)N-N-a_{k-1}\,,\infty) \;=\; \mathcal{S}_k[m-a_k,\infty)\,.$$

Hence, by Example 5.5, 5.9(ii) holds for k.

We come finally to the *proof of Lemma 5.12.*

Let $\mathcal{S} \subset \mathbf{D}(U_k)$ be the full subcategory with objects those $F \in \{C_k\}[m,\infty)$ for which the Lemma holds. We need to verify the conditions in Definition 5.1, i.e., we need to show:

(a) $C_k[-\ell] \in \mathcal{S}$ for all $\ell \ge m$; and

(b) for any $\mathbf{D}(U_k)$-triangle

$$F' \longrightarrow F \longrightarrow F'' \longrightarrow F'[1],$$

if $F', F'' \in \mathcal{S}$ then $F \in \mathcal{S}$.

For (a), we first use Lemma 5.11 to extend $\psi\colon C_k[-\ell] \to E|_{U_k}$ to a $\mathbf{D}(V_k)$-morphism $\phi\colon S_k[-\ell]\big|_{V_k} \to E|_{V_k}$. By Thomason's localization theorem, as formulated in [N2, p. 214, 2.1.5] (and further elucidated in [*ibid.,* p. 216, proof of Lemma 2.6]),[5] there is then a $\mathbf{D}_{\mathsf{qc}}(X)$-diagram, with top row a triangle of perfect complexes:

$$\begin{array}{ccccccc} \widetilde{P} & \longrightarrow & \widetilde{F}_1 & \xrightarrow{\;f\;} & S_k[-\ell] & \longrightarrow & \widetilde{P}[1] \\ & & {\scriptstyle g}\big\downarrow & & & & \\ & & E & & & & . \end{array}$$

and with $\widetilde{P}$ exact on V_k, so that $f|_{V_k}$ is an isomorphism; and furthermore,

$$\phi = (g|_{V_k}) \circ (f|_{V_k})^{-1}.$$

Since $S_k[-\ell] \in \mathcal{S}_k[\ell,\infty]$ (see Remark 5.4(i)), we need only show that we can choose $\widetilde{P} \in \mathcal{S}_{k-1}[\ell-N-a_{k-1}\,,\infty)$, because then we'll have

$$\widetilde{F}_1 \in \mathcal{S}_k[\ell-N-a_{k-1}\,,\infty) \subset \mathcal{S}_k[m-N-a_{k-1}\,,\infty).$$

The perfect complex $\widetilde{P}$ is exact outside $X - V_k = Y_{k-1}$, and we are assuming that 5.9(i) is true for $k-1$. It follows that there exists a triangle

$$P \longrightarrow \widetilde{P} \longrightarrow Q \longrightarrow P[1]$$

with $P \in \mathcal{S}_{k-1}[\ell-N-a_{k-1}\,,\infty)$ and $H^i(Q) = 0$ for all $i \ge \ell - N$. Since all the members of $\mathcal{S}_{k-1}$ are exact on V_k, the same is true of P (argue as in Remark 5.4(i)–(iv)).

[5]where in the absence of separatedness, $j_{\bullet *}$ should become $\mathbf{R}j_{\bullet *}$.

Now [N2, p. 58, 1.4.6] produces an octahedron on $P \to \widetilde{P} \to \widetilde{F}_1$, where the rows and columns are triangles:

$$
\begin{array}{ccccccc}
P & = & P & & & & \\
\downarrow & & \downarrow & & & & \\
\widetilde{P} & \longrightarrow & \widetilde{F}_1 & \xrightarrow{f} & S_k[-\ell] & \longrightarrow & \widetilde{P}[1] \\
\downarrow & & \downarrow{\scriptstyle\beta} & & \| & & \downarrow \\
Q & \longrightarrow & F' & \xrightarrow[f']{} & S_k[-\ell] & \xrightarrow[g]{} & Q[1] \\
\downarrow & & \downarrow{\scriptstyle h} & & & & \\
P[1] & = & P[1] & & & &
\end{array}
$$

Since $H^i\big(Q[1+\ell]\big) = 0$ for all $i \geq -N-1$, the definition of N (see Notation 5.8(vii)) forces the map $g\colon S_k[-\ell] \to Q[1]$ to vanish. The exact sequence

$$\mathrm{Hom}(S_k[-\ell], F') \xrightarrow{\text{via } f'} \mathrm{Hom}(S_k[-\ell], S_k[-\ell]) \xrightarrow{\text{via } g=0} \mathrm{Hom}(S_k[-\ell], Q[1])$$

shows there is a map $\iota\colon S_k[-\ell] \to F'$ with $f'\iota$ the identity map of $S_k[-\ell]$.

This gives rise to yet another octahedron, on $S_k[-\ell] \xrightarrow{\iota} F' \xrightarrow{h} P[1]$:

$$
\begin{array}{ccccccc}
P & = & P & & & & \\
\downarrow & & \downarrow & & & & \\
\widetilde{F} & \xrightarrow{\gamma} & \widetilde{F}_1 & \longrightarrow & Q & \longrightarrow & \widetilde{F}[1] \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \| & & \downarrow \\
S_k[-\ell] & \xrightarrow{\iota} & F' & \longrightarrow & Q & \longrightarrow & S_k[-\ell+1] \\
\downarrow & & \downarrow{\scriptstyle h} & & & & \\
P[1] & = & P[1] & & & &
\end{array}
$$

The first column is a triangle, with $\widetilde{F} \in \mathcal{S}_k[\ell - N - a_{k-1}\,, \infty)$, and $P|_{V_k}$ exact, so that $\alpha|_{V_k}$ is an isomorphism.

Moreover, if $\tilde{\psi}\colon \widetilde{F} \to E$ is the composite $\widetilde{F} \xrightarrow{\gamma} \widetilde{F}_1 \xrightarrow{g} E$, then

$$\alpha = f'\iota\alpha = f'\beta\gamma = f\gamma,$$

so that on V_k,

$$\phi\alpha = \phi f\gamma = g\gamma = \tilde{\psi},$$

proving (a).

Proof of (b).

Let $\psi\colon F\to E|_{U_k}$ be a $\mathbf{D}(U_k)$-morphism. Since $F'\in\mathcal{S}$, there exists a complex $\widetilde{F}'\in\mathcal{S}_k[m-N-a_{k-1}\,,\infty)$ and a $\mathbf{D}(X)$-morphism $\widetilde{\psi}'\colon\widetilde{F}'\to E$ whose restriction to U_k is isomorphic to the composite $F'\to F\xrightarrow{\psi}E|_{U_k}$. There results a triangle

$$\widetilde{F}'\xrightarrow{\ \widetilde{\psi'}\ }E\xrightarrow{\ \gamma'\ }E'\longrightarrow\widetilde{F}'[1]\ ,$$

and hence a commutative $\mathbf{D}(U_k)$-diagram (part of an octahedron):

$$\begin{array}{ccccccc}
F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & F'[1]\\
\| & & {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle g} & & \|\\
F' & \longrightarrow & E|_{U_k} & \xrightarrow{\gamma'|_{U_k}} & E'|_{U_k} & \longrightarrow & F'[1]\\
 & & {\scriptstyle\chi}\big\downarrow & & \big\downarrow{\scriptstyle h} & & \\
 & & G & =\!= & G & &
\end{array}$$

Since $F''\in\mathcal{S}$, there is an $\widetilde{F}''\in\mathcal{S}_k[m-N-a_{k-1}\,,\infty)$ and a $\mathbf{D}(X)$-morphism $\widetilde{\psi''}\colon\widetilde{F}''\to E'$ whose restriction to U_k is isomorphic to $g: F''\to E'|_{U_k}$. So there is a triangle

$$\widetilde{F}''\xrightarrow{\ \widetilde{\psi''}\ }E'\xrightarrow{\ \gamma''\ }E''\longrightarrow\widetilde{F}''[1]\ ;$$

whose restriction to U_k is isomorphic to

$$F''\xrightarrow{\ g\ }E'|_{U_k}\xrightarrow{\ h\ }G\longrightarrow F''[1]\ .$$

The restriction to U_k of the composite $E\xrightarrow{\gamma'}E'\xrightarrow{\gamma''}E''$ is isomorphic to the composite $\chi\colon E|_{U_k}\xrightarrow{\gamma'|_{U_k}}E'|_{U_k}\xrightarrow{h}G$. Completing $\gamma''\gamma'$ to a triangle

$$\widetilde{F}\xrightarrow{\ \tilde{\psi}\ }E\xrightarrow{\ \gamma''\gamma'\ }E''\longrightarrow\widetilde{F}[1]$$

and restricting to U_k, we obtain a triangle isomorphic to

$$F\xrightarrow{\ \psi\ }E|_{U_k}\xrightarrow{\ \chi\ }G\longrightarrow\widetilde{F}[1]\ .$$

That $\widetilde{F}\in\mathcal{S}_k[m-N-a_{k-1}\,,\infty)$ follows from Remark 5.4(v). □

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Joseph Lipman, Department of Mathematics, Purdue University, W. Lafayette, IN 47907, USA

E-mail address: `jlipman@purdue.edu`

Amnon Neeman, Centre for Mathematics and its Applications, Mathematical Sciences Institute, John Dednam Building, The Australian National University, Canberra, ACT 0200, Australia

E-mail address: `Amnon.Neeman@anu.edu.au`