Initial Ideals of Closed Determinantal Facet Ideals

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Notation

- $S = k[x_{i,j} \mid 1 \le i \le n, 1 \le j \le m]$ where $n \le m$ and k is any field.
- M is a generic $n \times m$ matrix of variables in S.
- $<$ denotes standard lexicographic order in S; that is, lexicographic order with $x_{11} < \cdots < x_{1m} < x_{21} < \cdots < x_{nm}$.
- $[\mathbf{b}] = [b_1, \dots, b_n]$ a maximal minor of M corresponding to columns $b_1, \dots, b_n$, where
$$1 \le b_1 < b_2 < \cdots < b_n \le m.$$
- Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$. Define $|\alpha| := \sum_{i=1}^n \alpha_i$ and $\alpha_{\le i} := (\alpha_1, \dots, \alpha_i)$, where $\alpha_{\le i} = \varnothing$ if $i \le 0$ and $\alpha_{\le i} = \alpha$ if $i \ge n$.

Preliminaries

- Assume throughout that Δ is a pure $(n-1)$-dimensional simplicial complex on m vertices.
- A **determinantal facet ideal** $J_\Delta \subseteq S$ is the ideal generated by determinants of the form $[\mathbf{b}]$ where $\mathbf{b}$ supports an $n-1$ face of Δ; that is, the columns of $[\mathbf{b}]$ correspond to the vertices of some facet $\sigma \in \Delta$.
- When $n = 2$, Δ corresponds to a graph G and J_G is called a **binomial edge ideal**.
- A simplicial complex Δ is said to be **closed** (with respect to a given labeling) if it satisfies any one of the following equivalent conditions:
  1. The minors generating the determinantal facet ideal J_Δ form a Gröbner basis with respect to lexicographic order.
  2. For any two facets $F = \{a_1 < \cdots < a_n\}$ and $G = \{b_1 < \cdots < b_n\}$ with $a_i = b_i$ for some i, the $(n-1)$-skeleton of the simplex on the vertex set $F \cup G$ is contained in Δ.
- The **clique complex** Δ^{clique} of Δ has as facets all maximal sets of vertices of Δ such that every possible choice of n vertices in a facet of Δ^{clique} is a face of Δ.

Betti Numbers of Initial Ideals

For any closed determinantal facet ideal, the initial ideal is of degree n and squarefree, so that Stanley-Reisner theory may be employed to compute the $\mathbb{Z}^{nm}$-graded Betti numbers.

Notation. Let $\text{in}(J_\Delta)$ denote the initial ideal with respect to $<$ of J_Δ.

Theorem (AV20)

Let Δ be a pure $(n-1)$-dimensional simplicial complex which is closed. Then the $\mathbb{Z}^{nm}$-graded Betti numbers of $\text{in}(J_\Delta)$ are either 0 or 1.

Idea of Proof. We study the Stanley-Reisner complex Γ of $\text{in}(J_\Delta)$. We observe that the restriction of Γ to monomials of certain forms is homotopy equivalent to a sphere, and that the restriction of Γ to any other monomial is contractible. By Hochster's formula, the result follows.

When is $\beta_{i,j}(J_\Delta) = \beta_{i,j}(\text{in}(J_\Delta))$?

Fact. Let I be a homogeneous ideal in a polynomial ring $k[x_1, \dots, x_n]$ over a field k. For any term order $<$, $\beta_{i,j}(I) \le \beta_{i,j}(\text{in}_<(I))$ for any i, j. However, it is rare for equality to hold.

Theorem (AV20)

Let Δ be a pure $(n-1)$-dimensional simplicial complex which is closed. When $n > 2$, the standard graded Betti numbers of J_Δ and $\text{in}(J_\Delta)$ coincide.

Idea of Proof. By [1], the Betti table of an ideal can always be obtained from the Betti table of one of its initial ideals via consecutive cancellations. By analyzing the nonzero Betti numbers of the initial ideal, we show that consecutive cancellations are never possible when $n > 2$.

Linear Strand of $\text{in}(J_\Delta)$

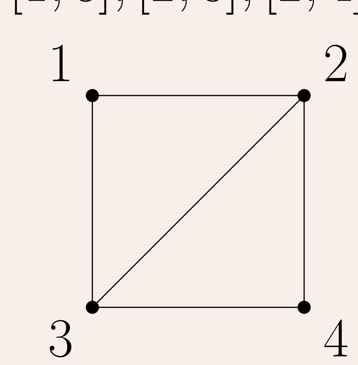
Definition. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| = \ell$ and $I = (i_1 < \cdots < i_{n+\ell})$. Define the indexing set
$$\mathcal{I}_<(\alpha, I) := \{(i, I_{i+j}) \mid i \in \{k \mid \alpha_k > 0\},\ |\alpha_{\le i-1}| \le j \le |\alpha_{\le i}|\}.$$

Definition. $\mathcal{C}_0^<(\Delta, M) := \bigwedge^n G$. For $i \ge 1$, let $\mathcal{C}_i^<(\Delta, M) \subseteq D_{i-1}(G^*) \otimes \bigwedge^{n+i-1} F$ denote the free submodule generated by all elements of the form
$$g^{*(\alpha)} \otimes f_\sigma,$$
where $\sigma \in \Delta^{\text{clique}}$ with $|\sigma| = n + i - 1$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| = i - 1$. Let $\mathcal{C}_\bullet^<(\Delta, M)$ denote the complex induced by the differentials
$$d_\ell(g^{*(\alpha)} \otimes f_\sigma) = \sum_{\{i \mid \alpha_i > 0\}} \sum_{(i,\sigma_j) \in \mathcal{I}_<(\alpha,\sigma)} (-1)^{j+1} x_{i\sigma_j} g^{*(\alpha - \epsilon_i)} \otimes f_{\sigma \setminus \sigma_j}$$
on the submodules defined above.

Theorem (AV20)

Assume that Δ is an $(n-1)$-pure closed simplicial complex. Let $F_\bullet$ denote the minimal graded free resolution of $\text{in}(J_\Delta)$ and let $F_\bullet^{\text{lin}}$ denote its linear strand; then
$$F_\bullet^{\text{lin}} \cong \mathcal{C}_\bullet^<(\Delta, M).$$

Example 1. Let G be the closed graph below, so $J_G = \{[1,2], [1,3], [2,3], [2,4], [3,4]\}$.



The clique complex of G has facets $\{1,2,3\}$ and $\{2,3,4\}$. Consider the basis element
$$g^{*(\alpha)} \otimes f_\sigma$$
where $\alpha = (0,1)$ and $\sigma = \{2,3,4\}$. Then
$$\mathcal{I}_<(\alpha, \sigma) = \{(2,3), (2,4)\}$$
so
$$d_2(g^{*(\alpha)} \otimes f_\sigma) = x_{2,3} f_{2,4} + x_{2,4} f_{2,3}.$$

When is $\beta_{i,j}(J_G) = \beta_{i,j}(\text{in}(J_G))$?

- It is still not known if, in general, the standard graded Betti numbers of closed binomial edge ideals and their initial ideals coincide. This was first conjectured in [2], where it is shown that they coincide for Cohen-Macaulay binomial edge ideals.
- For any closed graph G, it is known that the extremal betti numbers of J_G and $\text{in}(J_G)$ coincide by [3].
- Our work combined with [4] implies that the linear strands of J_G and $\text{in}(J_G)$ have the same graded Betti numbers.
- **Theorem (AV20).** Let G be the graph obtained by removing the edge $\{1, m\}$ from the complete graph on m vertices (as in Example 1). Then G is closed and $\beta_{i,j}(J_G) = \beta_{i,j}(\text{in}(J_G))$ for all i, j.

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