Containment and Bounds on Waldschmidt Constant of Ideals of Points

Sankhaneel Bisui & Thái Thành Nguyễn

Joint Work with Eloísa Grifo and Huy Tài Hà

Tulane University

Introduction

Take a set of points $X = \{P_1, \dots, P_n\} \subset \mathbb{P}^N_{\mathbb{K}},\ \mathbb{K} = \overline{\mathbb{K}}$, char $\mathbb{K} = 0$.
Question: What is the minimal degree $\alpha_m(X)$ of a hyper-surface which passes through the points with multiplicity at least m?

Conjectural Answer by Chudnovsky [3].

$$\frac{\alpha_m(X)}{m} \geq \frac{\alpha(X) + N - 1}{N},\ \alpha(X) = \alpha_1(X),\ \forall m \geq 1.$$

Symbolic Power. Take $I \subseteq R = \mathbb{K}[\mathbb{P}^N]$, then

$$I^{(m)} = \bigcap_{\mathfrak{p} \in \text{Ass}(I)} (I^m R_{\mathfrak{p}} \cap R).$$

Zariski-Nagata Theorem. Take $I \subseteq R = \mathbb{K}[\mathbb{P}^N]$ be radical, then

$$I^{(m)} = \{f \in \mathbb{K}[\mathbb{P}^N] : \partial_{\underline{\alpha}}(f) \in I \text{ for } |\underline{\alpha}| < m\}$$

and

$$I_X^{(m)} = \cap_{i=1}^{s} \mathfrak{p}_i^m \text{ where } I_X = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cdots \cap \mathfrak{p}_s.$$

Waldschmidt Constant.

$$\widehat{\alpha}(I_X) = \lim \frac{\alpha(I_X^{(m)})}{m} = \lim \frac{\alpha_m(X)}{m} = \inf \frac{\alpha(I_X^{(m)})}{m}$$

where $\alpha(J)$ is the minimal degree of generators of J.

Chudnovsky's Conjecture becomes *equivalent* to,

$$\widehat{\alpha}(I_X) \geq \frac{\alpha(I_X) + N - 1}{N}.$$

Known Results

Assume $\mathbb{K} = \overline{\mathbb{K}}$, char $\mathbb{K} = 0$.

- Any finite set of points in $\mathbb{P}^2_{\mathbb{K}}$ by Chudnovsky [3] also by Harbourne and Huneke [8].
- Esnault and Viehweg, 1983, showed,
$$\frac{\alpha(I_X^{(m)})}{m} \geq \frac{\alpha(I_X) + 1}{N}.$$
- Any finite set of general points in $\mathbb{P}^3_{\mathbb{K}}$ by Dumnicki [4].
- Any finite set of at most $N + 1$ general points in $\mathbb{P}^N_{\mathbb{K}}$ by Dumnicki [4].
- Any set of binomial number of points in $\mathbb{P}^N_{\mathbb{K}}$ forming a star configuration by Bocci and Harbourne [2].
- Any set of more than 2^N very general points in $\mathbb{P}^N_{\mathbb{K}}$ by Dumnicki and Tutaj-Gasińska [5].
- Any finite set of very general points in $\mathbb{P}^N_{\mathbb{K}}$. by Fouli, Mantero and Xie [6].

Containment

Appropriate Containment would imply Chudnovsky's Conjecture. Let $R = \mathbb{K}[\mathbb{P}^N],\ I \subset R$, *homogeneous radical with bigheight* $= h$.

Theorem 1 (Ein-Lazarsfeld-Smith, Hochster-Huneke, Ma-Schwede). *With I, h as above,*

$$I^{(hr)} \subseteq I^r, r \geq 1.$$

Conjecture 2 (Harbourne [8]). *With $I,\ h$ as above,*

$$I^{(hr-h+1)} \subseteq I^r, r \geq 1.$$

Conjecture 3 (Harbourne-Huneke [8]). *With I, h as above,*

(1) $I^{(hr-h+1)} \subseteq \mathfrak{m}^{(r-1)(h-1)} I^r,\ r \geq 1.$

(2) $I^{(hr)} \subseteq \mathfrak{m}^{r(h-1)} I^r, r \geq 1.$

Conjecture 3 Part 2 $\implies$ Chudnovsky's Conjecture.

Known Results:

B. Harbourne and C. Huneke [8] proved Conjecture 3 when $N = 2$.

Stable Containment

Stable Containment problem is asking whether Containment are true for all large value m or infinitely many m.

E.g Tohaneanu and Xie proved Stable Harbourne for very general points [9].

Appropriate Stable Containment $\implies$ Chudnovsky's Conjecture.

One approach to prove Stable Containment is the following:

Idea: *Containment (maybe stronger) for* ***one value*** *c would imply desired containment* ***for all*** *c* ***large*** *or for* ***infinitely many*** *c.*

Theorem 4 (Grifo [7]). *Let $I \subseteq \mathbb{K}[\mathbb{P}^N]$ be a radical ideal of big height h. If $I^{(hc-h)} \subseteq I^c$ for some constant $c \in \mathbb{N}$ then for all $r \gg 0$, we have*

$$I^{(hr-h)} \subseteq I^r.$$

More concretely, this containment holds for all $r \geqslant hc$.

Theorem 5 (Bisui-Grifo-Ha-Nguyễn [1]). *Let $I \subseteq \mathbb{K}[\mathbb{P}^N]$ be a radical ideal of big height h Suppose that for some value $c \in \mathbb{N}$, $I^{(hc-h)} \subseteq \mathfrak{m}^{c(h-1)} I^c$. For all $r \gg 0$, we have*

$$I^{(hr-h)} \subseteq \mathfrak{m}^{r(h-1)} I^r.$$

Example

Example 6 (Fermat configurations). $I = (x(y^n - z^n), y(z^n - x^n), z(x^n - y^n))$ in $\mathbb{K}[x, y, z]$ corresponds to a Fermat configuration of $n^2 + 2$ points in $\mathbb{P}^2$, where char$\mathbb{K} \neq 2$, $n \geqslant 3$ distinct n-th roots of unity. Although $I^{(3)} \not\subset I^2$, but

- $I^{(2r-1)} \subseteq \mathfrak{m}^{r-1} I^r$ for $r \gg 0$, and
- $I^{(2r)} \subseteq \mathfrak{m}^r I^r$ for $r \gg 0$.

Hence the stable containment hold for Fermat configuration.

Results for General Points

Theorem 7 (Bisui-Grifo-Ha-Nguyễn [1]). *Suppose that char $\mathbb{K} = 0$ and $N \geqslant 3$ and I defines a general set of s points in $\mathbb{P}^N_{\mathbb{K}}$,*

1. *The stable containment*
$$I^{(Nr-N)} \subseteq I^r$$
holds for $r \gg 0$.
2. *If $s \geqslant 4^N$ then the stable containment*
$$I^{(Nr)} \subseteq \mathfrak{m}^{r(N-1)} I^r$$
holds for $r \gg 0$.
3. *If $s \geqslant \binom{N^2+N}{N}$ then the stable containment*
$$I^{(Nr-N+1)} \subseteq \mathfrak{m}^{(r-1)(N-1)} I^r$$
holds for $r \gg 0$.

Theorem 8 (Bisui-Grifo-Ha-Nguyễn [1]). *Suppose that char $\mathbb{K} = 0$, and $N \geq 3$,*

1. *Suppose that $s \geqslant 4^N$. Chudnovsky's Conjecture holds for a general set of s points in $\mathbb{P}^N_{\mathbb{K}}$.*
2. *. Let J be the defining ideal of a fat point scheme $X = tP_1 + \cdots + tP_s$ in $\mathbb{P}^N_{\mathbb{K}}$ with a general support, for some $t \in \mathbb{N}$. If $t \geq 2$ then Chudnovsky's Conjecture holds for J, i.e.,*
$$\hat{\alpha}(J) \geq \frac{\alpha(J) + N - 1}{N}.$$

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