$\mathfrak{m}$-ADIC PERTURBATIONS IN NOETHERIAN LOCAL RINGS

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1. INTRODUCTION

Throughout, $(R, \mathfrak{m}_R)$ is a complete local Noetherian ring, and $I \subset \mathfrak{m}_R$ is a parameter ideal.

Defintion. *A $\mathfrak{m}_R$-adic* **perturbation** *of I, of* **order** *$T \in \mathbb{N}$, is an ideal $(f_1+\varepsilon_1, \dots, f_c+\varepsilon_c) = \left(\underline{f+\varepsilon}\right) \subset R$, where $(f_1, \dots, f_c) = I$ are minimal generators, and $\varepsilon_1, \dots, \varepsilon_c \in \mathfrak{m}_R^T$. A perturbation, $\left(\underline{f+\varepsilon}\right)$, is 'small,' if $\varepsilon_1, \dots, \varepsilon_c \in \mathfrak{m}_R^T$ with T 'large'.*

The relationship between the quotient, R/I, and the rings $R/\left(\underline{f+\varepsilon}\right)$, where $\left(\underline{f+\varepsilon}\right)$ is a small perturbation of I, is a subject of active research [Eis74, SS20, MQS19, PS18, ST96, QT20]. In [SS20] Smirnov and De Stefani show that, in certain cases of interest, this question is related to the 'deformation' of properties. The behavior of the Hilbert function and associated graded rings under small perturbations has been investigated by several authors (see [ST96], and [MQS19] for recent breakthroughs).

In [PS18] Polstra and Smirnov prove that the Hilbert-Kunz multiplicity enjoys a remarkable kind of continuity with respect to $\mathfrak{m}_R$-adic perturbations when $(R, \mathfrak{m}_R)$ is Cohen-Macaulay and F-finite, and $I \subset R$ is a parameter ideal such that R/I is reduced and equidimensional.

In lemmas 1 and 2 we establish techniques that apply to perturbations of parameter ideals in an arbitrary complete Noetherian local ring. These results show that, if M is any finite R-module satisfying certain technical conditions (see (i) of lemma 2), and $\left(\underline{f+\varepsilon}\right)$ is a small perturbation of a parameter ideal $I \subset R$, M/IM and $M/\left(\underline{f+\varepsilon}\right)M$ are very strongly correlated.

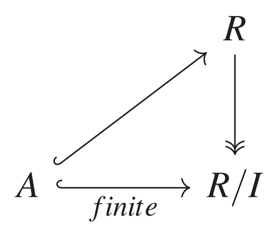
Using these tools, we establish new results about the behavior of Hilbert-Samuel and Hilbert-Kunz multiplicities under small perturbations.

2. TECHNICAL RESULTS AND SETUP

The results of this segment are formulated in terms of a particular diagram.
(Setup):

Given complete Noetherian local rings $(R, \mathfrak{m}_R, \kappa)$ and $(A, \mathfrak{m}_A)$, and $I \subset R$ a parameter ideal, D denotes the following diagram of local ring maps:

D:



Where $R \twoheadrightarrow R/I$ is the quotient map and $A \hookrightarrow R/I$ is a module finite extension. Note that, when D commutes, the ideal $I + \mathfrak{m}_A R \subset R$ is $\mathfrak{m}_R$-primary.

For us, this setup is achieved as follows: a choice of parameters, $x_1, \dots, x_c$, on R/I induces a diagram of this form, where $A = \kappa[[x_1, \dots, x_c]]$, and $A \hookrightarrow R$ is a lift of the module finite Cohen extension $A \hookrightarrow R/I$.

Lemma 1. *Suppose $(R, \mathfrak{m}_R)$, $(A, \mathfrak{m}_A)$ are complete Noetherian local rings, $I \subset R$ is a parameter ideal, and D is a commuting diagram of local rings and local ring maps, as in the setup above.*
Let $T \in \mathbb{N}$ be large enough so that $\mathfrak{m}_R^T \subset \mathfrak{m}_R(I + \mathfrak{m}_A R)$.
Then, for any $(f_1, \dots, f_c) = I$ and any $\varepsilon_1, \dots, \varepsilon_c \in \mathfrak{m}_R^T$:

(a) the composition

$$A \hookrightarrow R \to \frac{R}{\left(\underline{f+\varepsilon}\right)}$$

is module finite.

(b) if M is any finite R-module, any elements $m_1, \dots, m_k \in M$ that map to minimal generators for M/IM over A, also map to minimal generators for $M/\left(\underline{f+\varepsilon}\right)M$ as an A-module.

Lemma 2. *Assume the setup of lemma 1, that M is a finite R-module, and suppose that the images of $v_1, \dots, v_n \in M$ in M/IM span a free A-module of maximal rank. Suppose, in addition that*

(i) The localization $M_\mathfrak{q}$ is Cohen-Macaulay for each prime $\mathfrak{q} \in Spec(R)$ with $\mathfrak{q} \supset I$ and $\dim R/\mathfrak{q} = \dim R/I$.

(ii) $depth_{\mathfrak{m}_A}(A) \geq 1$, so there is an $x \in \mathfrak{m}_A$ which is a nzd.

Then, there is a $T \in \mathbb{N}$ such that, for all minimal generators $(f_1, \dots, f_c) = I$, and every $\varepsilon_1, \dots, \varepsilon_c \in \mathfrak{m}_R^T$, the images of $v_1, \dots, v_n$ span a free A-module in $M/\left(\underline{f+\varepsilon}\right)M$ of maximal rank.

3. HILBERT-SAMUEL MULTIPLICITY

Corollary 1. *When the conditions of lemma 2 are satisfied, and A is a domain, there is an equality,*

$$rank_A\left(M/\left(\underline{f+\varepsilon}\right)M\right) = rank_A(M/IM),$$

for all $\mathfrak{m}_R$-adic perturbations of I of sufficiently high order.

Recall, after possibly passing to a faithfully flat extension with infinite residue field (see section 8.4 of [HSPS06]), we can chose parameters on R/I that generate a minimal reduction of the maximal ideal. Letting $A \hookrightarrow R/I$ be a corresponding Cohen extension, there is an equality between Hilbert-Samuel multiplicity and A-rank.

Combining this with corollary 1 and some easy results about reductions, we prove the following theorem, which is a weaker conclusion than theorem 3.7 of [MQS19], but applies to more general perturbations:

Theorem 1. *Suppose that $I \subset R$ is a parameter ideal in a complete local Noetherian ring, $(R, \mathfrak{m}_R)$, and M is a finite R-module. Assume that $M_\mathfrak{q}$ is CM for every $\mathfrak{q} \in Spec(R)$ such that $\mathfrak{q} \supset I$ and $\dim R/\mathfrak{q} = \dim R/I$. Then, there is a $T \in \mathbb{N}$ such that for all minimal generators $(f_1, \dots, f_c) = I$, and every $\varepsilon_1, \dots, \varepsilon_c \in \mathfrak{m}_R^T$, there is an equality*

$$e(M/IM) = e\left(M/\left(\underline{f+\varepsilon}\right)M\right)$$

4. HILBERT-KUNZ MULTIPLICITY

Applying lemma 2 to R, we see that, if $r_1, \dots, r_n \in R$ generate a free A-submodule of R/I of maximal rank, then they will do the same for $R/\left(\underline{f+\varepsilon}\right)$, when $\left(\underline{f+\varepsilon}\right)$ is a sufficiently small perturbation of I.

This allows us to show, when A is a normal domain, that the discriminants, $D_A\left(R/\left(\underline{f+\varepsilon}\right)\right)$, can be made arbitrarily $\mathfrak{m}_A$-adically close (in A). Combining this reasoning with corollary 2.5 of [Smi19], we are able to retrace the argument of section 3.1 in [PS18] without assuming R is CM. The result generalizes corollary 3.7 of [PS18]:

Theorem 2. *Suppose that $(R, \mathfrak{m}_R)$ is a complete, local, F-finite ring, and that $I \subset R$ is a parameter ideal such that R/I is equidimensional.*
Assume that, for every $\mathfrak{q} \in Spec(R)$ such that $\mathfrak{q} \supset I$ and $\dim R/\mathfrak{q} = \dim R/I$

(i) $R_\mathfrak{q}$ is Cohen-Macaulay, and

(ii) $(R/I)_\mathfrak{q}$ is reduced

Then, for every $\delta > 0$, there is a $T \in \mathbb{N}$ such that

$$\left|e_{HK}(R/I) - e_{HK}\left(R/\left(\underline{f+\varepsilon}\right)\right)\right| < \delta$$

for all minimal generators $(f_1, \dots, f_c) = I$, and every $\varepsilon_1, \dots, \varepsilon_c \in \mathfrak{m}_R^T$.

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