Hilbert-Kunz function of the Rees algebra

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Abstract

We prove that the Hilbert-Kunz function of the ideal $(J, It)\mathcal{R}(I)$, where J is an $\mathfrak{m}$-primary ideal and I is a parameter ideal of a 1-dimensional local ring $(R, \mathfrak{m})$, is a quasi-polynomial for large values.

For $s \in \mathbb{N}$, we compute the Hilbert-Samuel function of the R-module $I^{[s]}$ and obtain an explicit description of the generalized Hilbert-Kunz function of the ideal $(I, It)\mathcal{R}(I)$ when I is a parameter ideal in a Cohen-Macaulay local ring of dimension $d \geq 2$, proving that the generalized Hilbert-Kunz function is a piecewise polynomial in this case.

Preliminaries

Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring of prime characteristic $p > 0$ and let I be an $\mathfrak{m}$-primary ideal.

- Put $q = p^e$ for some $e \in \mathbb{N}$. The q^{th}-Frobenius power of I is the ideal $I^{[q]} = (x^q \mid x \in I)$.
- The function $e \mapsto \ell_R(R/I^{[p^e]})$ is called the **Hilbert-Kunz function** of R with respect to I.
- P. Monsky proved that

$$\ell_R(R/I^{[q]}) = e_{HK}(I, R)q^d + O(q^{d-1}),$$

where $e_{HK}(I, R)$ is a positive real number called the **Hilbert–Kunz multiplicity** of R with respect to I. We write $e_{HK}(R) := e_{HK}(\mathfrak{m}, R)$ and $e_{HK}(I) := e_{HK}(I, R)$.

- The **Hilbert-Samuel function** of R with respect to I is defined as $H_I(n) := \ell_R(R/I^n)$. It is a polynomial function of n of degree d. In particular, there exists a polynomial $P_I(x) \in \mathbb{Q}[x]$ such that $H_I(n) = P_I(n)$ for all large n.
- The **Hilbert-Samuel multiplicity** of R with respect to I is defined as

$$e(I, R) := \lim_{n \to \infty} \frac{\ell_R(R/I^n)}{n^d/d!}.$$

We write $e(I) := e(I, R)$ and $e(R) := e(\mathfrak{m}, R)$.

- A **quasi-polynomial** of degree d is a function $f : \mathbb{Z} \to \mathbb{C}$ of the form

$$f(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \cdots + c_0(n),$$

where each $c_i(n)$ is a periodic function and $c_d(n)$ is not identically zero.

- Define $a_i(R) := \max\{u \in \mathbb{Z} \mid \mathrm{H}^i_{\mathfrak{m}}(R)_u \neq 0\}$ if $\mathrm{H}^i_{\mathfrak{m}}(R) \neq 0$, and $-\infty$ otherwise.

Known results

- **(P. Monsky)** In the case of 1-dimensional rings, the Hilbert-Kunz function

$$\ell_R(R/I^{[p^e]}) = e_{HK}(I, R)q + \delta_e,$$

where δ_e is a periodic function of e, for large e.

- Let $\mathcal{R}(I) = \oplus_{n \geq 0} I^n t^n$ denote the Rees algebra of I.
- **(K. Eto - K.-i. Yoshida)** Put $c(d) = (d/2) + d/(d+1)!$. Then

$$e_{HK}(\mathcal{R}(I)) \leq c(d) \cdot e(I).$$

Moreover, equality holds if and only if $e_{HK}(R) = e(I)$.

Joint work with Mitra Koley and Jugal K. Verma

Dimension 1

Theorem 1. *Let $(R, \mathfrak{m})$ be a 1-dimensional Noetherian local ring with prime characteristic $p > 0$. Let I, J be $\mathfrak{m}$-primary ideals. Then*

$$e_{HK}((J, It)\mathcal{R}(I)) = e(J).$$

Theorem 2. *Let $(R, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay local ring with prime characteristic $p > 0$. Let I be a parameter ideal and J be an $\mathfrak{m}$-primary ideal. Then for $q = p^e$, where $e \in \mathbb{N}$ is large,*

$$\ell_R\left(\frac{\mathcal{R}(I)}{(J, It)^{[q]}}\right) = e(J)q^2 + \alpha_J(e)q,$$

where $\alpha_J(e) = \ell_R(R/J^{[q]}) - e(J)q$ is a periodic function of e.

Example 3. Let $R = k[[X, Y]]/(X^5 - Y^5)$, where k is a field of prime characteristic $p \equiv \pm 2 (\mathrm{mod}\ 5)$. Let $\mathfrak{m}$ be the maximal ideal of R and $I = (x)$. Put $q = p^e$, for some $e \in \mathbb{N}$. Using Theorem 2, we get

$$\ell_R\left(\frac{\mathcal{R}(I)}{(\mathfrak{m}, It)^{[q]}}\right) = e(\mathfrak{m})q^2 + \alpha_{\mathfrak{m}}(e)q$$

$$= 5q^2 + \begin{cases} -4q & \text{if } e \text{ is even} \\ -6q & \text{if } e \text{ is odd.} \end{cases}$$

Generalized HK-function

(A. Conca) For $s \in \mathbb{N}$, let $I^{[s]} = (a_1^s, \ldots, a_n^s)$ where $\{a_1, a_2, \ldots, a_n\}$ is a fixed set of generators of I. The **generalized Hilbert-Kunz function** is defined as

$$HK_{R,I}(s) := \ell_R\left(\frac{R}{I^{[s]}}\right).$$

The **generalized Hilbert-Kunz multiplicity** is defined as $\lim_{s \to \infty} HK_{R,I}(s)/s^d$, whenever the limit exists.

Theorem 4. *Let R be a Cohen-Macaulay local ring of dimension $d \geq 2$ and let I be a parameter ideal. For a fixed $s \in \mathbb{N}$, $\ell_R(I^{[s]}/I^{[s]}I^n)$ equals*

- $d \cdot H_I(n)$, *if $1 \leq n \leq s$,*
- $\sum_{i=1}^{d-1} (-1)^{i+1} \binom{d}{i} H_I(n - (i-1)s)$, *if $s + 1 \leq n \leq (d-1)s - 1$,*
- $H_I(n + s) - s^d e(I)$, *if $n \geq (d-1)s$.*

Theorem 5. *Let R be a Cohen-Macaulay local ring of dimension $d \geq 2$. Let I be a parameter ideal of R. Let $s \in \mathbb{N}$. Then*

$$\ell_R\left(\frac{\mathcal{R}(I)}{(I, It)^{[s]}}\right)$$

is a polynomial in s, for all $s \geq d$. Moreover, the generalized Hilbert-Kunz multiplicity $e_{HK}((I, It)\mathcal{R}(I)) = c(d) \cdot e(I)$.

Example 6. Let $R = k[[X, Y, Z]]/(XY - Z^n)$, for some positive integer $n \geq 2$ and let I be a parameter ideal of R. Then for $s \geq 2$,

$$\ell_R\left(\frac{\mathcal{R}(I)}{(I, It)^{[s]}}\right) = e(I)\left[\frac{4}{3}s^3 - \frac{1}{3}s\right].$$

Joint work with Arindam Banerjee and Jugal K. Verma

Stanley-Reisner rings

Let $S = k[x_1, \ldots, x_r]$ be a polynomial ring in r variables over a field k and let $\mathfrak{m}$ be the maximal homogeneous ideal of S. Let $P_1, \ldots, P_\alpha$, for $\alpha \geq 2$, be distinct S-ideals generated by subsets of $\{x_1, \ldots, x_r\}$. Let $I = \cap_{i=1}^{\alpha} P_i$ and $R = S/I$. Suppose $\mathfrak{n} = \mathfrak{m}/I$ denotes the maximal homogeneous ideal of R and $\dim(R) = d$.

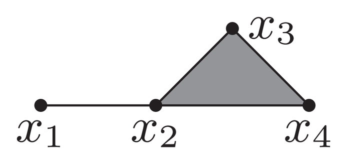
Theorem 7. *Set $\delta = \max\{|a_i(R)| \colon a_i(R) \neq -\infty\}$. Then for $s > \delta$,*

$$\ell\left(\frac{\mathcal{R}(\mathfrak{n})}{(\mathfrak{n}, \mathfrak{n}t)^{[s]}}\right)$$

is a polynomial in s of degree $d + 1$.

Examples

Example 8. Let Δ be the simplicial complex

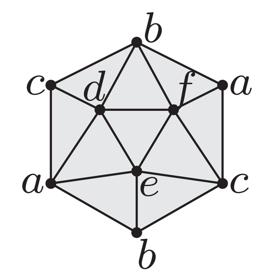


Then $R = k[x_1, x_2, x_3, x_4]/((x_1) \cap (x_3, x_4))$ is the Stanley-Reisner ring of Δ, with f-vector $f(\Delta) = (1, 4, 4, 1)$ and h-vector $h(\Delta) = (1, 1, -1, 0)$. For all $s > 3$,

$$\ell\left(\frac{\mathcal{R}(\mathfrak{n})}{(\mathfrak{n}, \mathfrak{n}t)^{[s]}}\right) = \frac{13}{8}s^4 + \frac{13}{12}s^3 - \frac{9}{8}s^2 - \frac{7}{12}s.$$

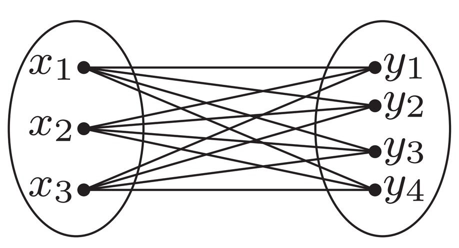
Example 9 (Triangulation of real projective plane). Let Δ be the triangulation of the real projective plane. Let R be the corresponding Stanley-Reisner ring of Δ. The f-vector of R is $f(\Delta) = (1, 6, 15, 10)$ and h-vector of R is $h(\Delta) = (1, 3, 6, 0)$. For $s \geq 1$,

$$\ell\left(\frac{\mathcal{R}(\mathfrak{n})}{(\mathfrak{n}, \mathfrak{n}t)^{[s]}}\right) =$$



$$390\binom{s+3}{4} - 720\binom{s+2}{3} + 372\binom{s+1}{2} - 41s.$$

Example 10. Let Δ be a complete bipartite graph $K_{3,4}$. Let $S = k[x_1, x_2, x_3, y_1, y_2, y_3, y_4]$.



The edge ideal of $K_{3,4}$ is the ideal $I = (x_i y_j \mid 1 \leq i \leq 3, 1 \leq j \leq 4)$. Let $R = S/I$. For all $s > 4$,

$$\ell\left(\frac{\mathcal{R}(\mathfrak{n})}{(\mathfrak{n}, \mathfrak{n}t)^{[s]}}\right) = \frac{61}{30}s^5 + \frac{19}{24}s^4 - \frac{1}{12}s^3 - \frac{7}{24}s^2 - \frac{9}{20}s - 1.$$

References

1. Aldo Conca. Hilbert-Kunz function of monomial ideals and binomial hypersurfaces. *Manuscripta Math.*, 90(3):287-300, 1996.

2. Kazufumi Eto and Ken-ichi Yoshida. Notes on Hilbert-Kunz multiplicity of Rees algebras. *Comm. Algebra*, 31(12):5943-5976, 2003.

3. Paul Monsky. The Hilbert-Kunz function. *Math. Ann.*, 263(1):43-49, 1983.