Reducedness of formally Unramified Algebras over fields

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A Natural Question

- An algebra over a given field is called *formally unramified* if the corresponding module of Kahler differentials is zero.
- **A Natural Question :** *Are formally unramified algebras over a field reduced?*

Examples with a Negative Answer

- The answer to the **A Natural Question** is negative in general. Examples are given below.
- **Example 1:** It is easier to construct examples in positive characteristics. The Frobenius endomorphism of $\mathbb{F}_p[X, X^{1/p}, \dots, X^{1/p^n}, \dots]/(X)$ is surjective, so the algebra is unramified over $\mathbb{F}_p$; but not reduced.
- **Example 2:** The construction of an unramified, non-reduced algebra over a characteristic zero field is more involved. We sketch one construction which was originally proposed by Ofer Gabber.
  Step 1: Given a char 0 field k, start with an Artinian local k-algebra (R_0, m_0, k) and a non-zero $f \in R_0$ such that $f^2 = 0$ and $df = 0 \in \Omega_{R_0/k}$. One can take, for example, $R_0 = k[[X, Y]]/(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y})$, where $F = X^5 + Y^5 + X^2Y^2$ and f to be the image of F in R_0.
  Step 2: Set $B_t = \underbrace{R_0 \otimes_k \dots \otimes_k R_0}$-(t-1) times, $f_t = f \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes \dots \otimes 1 \otimes f \in B_t$. Given $g \in R_0$ such that $g^{s-1} \neq 0; g^s = 0$, take $R' = R_0 \otimes_k B_s/(g \otimes 1 - 1 \otimes f_s)$. Then the natural map $R_0 \to R'$ is injective and induced map $\Omega_{R_0/k} \to \Omega_{R'/k}$ sends $d(g)$ to zero. Since R, R' are both finite dimensional k-vector spaces, by repeating the same process, one can construct an Artinian local k-algebra extension (R_1, m_1, k) of R_0 such that the induced map $\Omega_{R_0/k} \to \Omega_{R_1/k}$ is zero.
  Step 3: By repeating the process in **Step 2**, construct a countable chain of local Artinian k-algebras $(R_0, m_0, k) \subseteq (R_1, m_1, k) \subseteq \dots$ such that the induced maps $\Omega_{R_i/k} \to \Omega_{R_{i+1}/k}$ are all zero. Then $\bigcup_{n \in \mathbb{N}} R_n$ is unramified over k but not reduced.

Affirmative Answer in the Local Case

- The following few results demonstrate that if the unramified algebra is local and 'nicer', it must be reduced.
- **Theorem 1:** *Let (R, m) be a local algebra (not necessarily Noetherian) over a field k such that the field extension $k \subseteq R/m$ is separable (the field extension need not be even finitely generated). Suppose that $\bigcap_{n \in \mathbb{N}} m^n = 0$. If $\Omega_{R/k} = 0$, R must be a field.*
  Sketch of Proof: Set $R' = R/m^2$, $L = R/m$. Since $k \hookrightarrow L$ is formally smooth (since $k \subseteq R/m$ is separable), using formal lifting property one can show that there is a k-algebra map $L \to R'$ such that the composition $L \to R' \to R'/m$ is an isomorphism. Thus we have an isomorphism
  $$\frac{m}{m^2} \to \Omega_{R'/L} \otimes_{R'} L$$
  Now $\Omega_{R'/k} = 0$ implies $\Omega_{R'/L} = 0$. Thus $m = m^2$. This implies $m = m^n$ for all $n \in \mathbb{N}$. From our assumption $\bigcap_{n \in \mathbb{N}} m^n = 0$, it follows that $m = 0$.
- **Corollary 1:** Any Noethrian algebra which is formally unramified over a perfect field (any field k such that char(k)=0; or char(k)=$p > 0$ and $k = k^p$) is product of fields- hence reduced

Affirmative Answer in the Graded Case

- **Theorem 2:** *Fix a perfect field k. Let R be an $\mathbb{N}$-graded algebra whose degree zero piece R_0 is Noetherian (we are not assuming k is inside the degree zero piece). If $\Omega_{R/k} = 0$, $R = R_0$ and R is reduced.*
- **Proof:**
  - This is a corollary of **Theorem 1**.
  - When $k \subseteq R_0$, the theorem also follows from **Theorem 3** where we carefully study the kernel of the universal derivation of an $\mathbb{N}$-graded algebra.
- **Theorem 3:** *Let R be an $\mathbb{N}$-graded ring containing a field k.*
  i. *If k has characteristic zero, then the kernel of the universal derivation $d : R \to \Omega_{R/R_0}$ is R_0.*
  ii. *If k has characteristic $p > 0$, then the kernel of the universal derivation $d : R \to \Omega_{R/R_0}$ is contained in the p-th Veronese subring of R:*
  $$\ker(d : R \to \Omega_{R/R_0}) \subseteq \bigoplus_{j \in \mathbb{N}} R_{jp}.$$

More Examples

- **Corollary 1** *raises the natural question: is a formally unramified Noetherian algebra over a field always reduced?*
  The following example shows that the answer to the above question is negative.
- **Example 3:** Fix $L = \mathbb{F}_p(x)$ and let $k = \mathbb{F}_p(x^{\frac{1}{p^\infty}})$ be the perfection of L. We will construct a Noetherian local k-algebra which is formally unramified over L but is not reduced.
  For $f(x) \in L$, let $f'(x)$ denote the derivative of $f(x)$ with respect to x. Set $A = k[Z]/(Z^2)$, and let z be the image of $Z \in k[Z]$ in A. Consider the additive map $\phi : L \to A$ given by $\phi(f(x)) = f(x) + f'(x)z$. It is not hard to verify that ϕ is a ring homomorphism, using the fact that $z^2 = 0$. View A as an L-algebra using this map (note: we are *not* using the "obvious" L structure induced by the inclusions $L \subset k \subset A$). Clearly A is a non-reduced Noetherian local L-algebra.
  We now verify that $\Omega_{A/L} = 0$. For this, it suffices to check that the differential da is zero in $\Omega_{A/L}$ for each $a \in A$. Since k is perfect, we can write $a = g_1^p + g_2^p z$, for some $g_1, g_2 \in k$. So $da = g_2^p dz$. Now, dz is zero since $dz = d((x^{1/p})^p + z) = d(\phi(x))$.

Questions

- The module of Kahler differentials is the zeroth cohomology of the relative cotangent complex. So, it's natural to ask the following question, to which we do not have an answer. This question appears as Question C.3, (ii), in [3], where it is attributed to Bhargav Bhatt (also see [2]).
- **Question:** Let A be a $\mathbb{Q}$-algebra such that the cotangent complex $\mathbb{L}_{A/\mathbb{Q}}$ is quasi-isomorphic to the zero complex. Is A reduced?

References

[1] Karen E. Smith Alapan Mukhopadhyay. *Reducedness of Formally Unramified Algebras over Fields*. URL: https://arxiv.org/abs/2005.05833.
[2] Bhargav Bhatt. *An Imperfect Ring With a Trivial Cotangent Complex*. URL: http://www-personal.umich.edu/~bhattb/math/trivial-cc.pdf.
[3] Matthew Morrow. *Topological Hochschild homology in arithmetic geometry*. URL: http://swc.math.arizona.edu/aws/2019/2019MorrowNotes.pdf.

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