Birational Geometry through Zariski-Riemann spaces

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Setup and program

Given a finitely generated field extension K/k of transcendence degree $d \geqslant 1$ (with k algebraically closed field), **our goal is to study the birational geometry of K/k** (i.e., the collection of all varieties defined over k and having function field K) **using the theory of valuations**. More specifically we focus on the following set, known as the **Zariski-Riemann space of** K/k:

$$\mathfrak{ZR}(K|k) = \{V \text{ valuation ring} \mid k \subseteq V,\ \mathrm{Frac}(V) = K\}.$$

We consider $\mathfrak{ZR}(K|k)$ as a topological space equipped with the **Zariski topology**, having as basic open subsets the sets of the form

$$\mathfrak{u}(f_1, \dots, f_s) := \{V \mid f_1, \dots, f_s \in V\},$$

where the f_i range in K.

A brief historic note

In the case of curves ($d = 1$) $\mathfrak{ZR}(K|k)$ corresponds to the unique nonsingular projective model of K/k (see [1, Chapter I.6]).

In the case $d > 1$, Zariski-Riemann spaces were introduced in [3] by Oscar Zariski who used them to define and study local uniformization in characteristic 0. He proved that the space $\mathfrak{ZR}(K|k)$ is compact and used it to solve the resolution of singularities problem for surfaces in [2] and for 3-folds in [4].

References

[1] Robin Hartshorne. *Algebraic Geometry.* Graduate Texts in Mathematics, 52. New York: Springer-Verlag, 1997.

[2] Oscar Zariski. "A simplified proof for the resolution of singularities of an algebraic surface". In: *Annals of Mathematics* 43.3 (1942), pp. 583–593.

[3] Oscar Zariski. "Local uniformization of algebraic varieties". In: *Annals of Mathematics* 41.4 (1940), pp. 852–896.

[4] Oscar Zariski. "The compactness of the Riemann manifold of an abstract field of algebraic functions". In: *Bulletin of the American Mathematical Society* 50.10 (1944), pp. 683–691.

From algebra to geometry

The connection between valuation theory and the geometric world of schemes is classically achieved by using the following steps:

1. Define a sheaf $\mathcal{O}$ on $\mathfrak{ZR}(K|k)$ by the rule $\mathcal{O}(\mathfrak{U}) := \bigcap_{V \in \mathfrak{U}} V$, for any non-empty open $\mathfrak{U} \subseteq \mathfrak{ZR}(K|k)$.
2. For any such set obtain a normal affine scheme $\underline{\mathtt{sch}}(\mathfrak{U}) := \mathrm{Spec}(\mathcal{O}(\mathfrak{U}))$.
3. Restrict the attention to those basic open subsets $\mathfrak{U}$ of $\mathfrak{ZR}(K|k)$ such that $\mathrm{Frac}(\mathcal{O}(\mathfrak{U})) = K$, called **charts**.
4. For any inclusion of charts $\mathfrak{U} \subseteq \mathfrak{V}$, obtain a canonical birational morphism $\underline{\mathtt{sch}}(\mathfrak{U}) \to \underline{\mathtt{sch}}(\mathfrak{V})$ of affine normal k-varieties.

Finding the right categories

Stemming from these classical thoughts, we try to utilize the same framework to construct more general varieties. We define an **atlas** $\mathcal{U}$ to be a finite collection of charts which forms a covering of $\mathfrak{ZR}(K|k)$ and such that the birational morphisms induced by each inclusion $\mathfrak{U} \cap \mathfrak{V} \subseteq \mathfrak{U}$ (for $\mathfrak{U}, \mathfrak{V} \in \mathcal{U}$) is an open immersion. On one hand, on the algebraic side, we take the category $\underline{\mathtt{ATLAS}}(K|k)$ whose objects are atlases of $\mathfrak{ZR}(K|k)$ and whose morphisms are given by some specific kind of refinements.

On the other hand, on the geometric side, we consider the category $\underline{\mathtt{PROPNORM}}(K|k)$ of all proper, normal k-varieties with function field K together with birational morphisms.

Main Theorem

Let K/k be a finitely generated field extension such that k is algebraically closed. Then there is an equivalence of categories

$$\underline{\mathtt{sch}} \colon \underline{\mathtt{ATLAS}}(K|k) \to \underline{\mathtt{PROPNORM}}(K|k).$$

Sketch of the proof

- Given an atlas $\mathcal{U} = \{\mathfrak{U}_1, \dots, \mathfrak{U}_n\}$ we take the affine normal schemes $\underline{\mathtt{sch}}(\mathfrak{U}_i)$, for $i = 1, \dots, n$.
- Thanks to the property defining atlases, we can glue these schemes together to obtain a normal k-variety $\underline{\mathtt{sch}}(\mathcal{U})$.
- This variety is proper because of the valuative criterion for properness (in the form of [1, II. Ex 4.5(c)]).

By tweaking the way we define morphisms in the category $\underline{\mathtt{ATLAS}}(K|k)$ it is possible to work more generally with proper, normal k-varieties and birational maps (rather than birational morphisms).

This theorem exhorts a plan aimed at regaining the main constructions of birational geometry through the lens of valuation rings and atlases.

Projective varieties

Example: Let $K = k(x, y)$ and consider the atlas

$$\mathcal{S} := \{\mathfrak{S}_1 = \mathfrak{u}(x, y), \mathfrak{S}_2 = \mathfrak{u}(\tfrac{1}{y}, \tfrac{x}{y}), \mathfrak{S}_3 = \mathfrak{u}(\tfrac{1}{x}, \tfrac{y}{x})\}.$$

Using the identifications $x = X/Z$, $y = Y/Z$, these three charts correspond to the standard open covering of $\mathbb{P}^2_k = \mathrm{Proj}\,[X, Y, Z]$ and therefore $\underline{\mathtt{sch}}(\mathcal{S}) = \mathbb{P}^2_k$.

More generally, if $K = k(x_1, \dots, x_d)$, we set $x_0 := 1$ and we consider the atlas

$$\mathcal{S}_d := \{\mathfrak{S}_i = \mathfrak{u}(\tfrac{x_0}{x_i}, \dots, \tfrac{x_d}{x_i}) \mid i = 0, \dots, d\},$$

then $\underline{\mathtt{sch}}(\mathcal{S}_d) = \mathbb{P}^d_k$. We call $\mathcal{S}_d$ the **standard atlas** defined by the tuple $\mathbf{x} = (x_1, \dots, x_d)$.

It is possible to prove that for an arbitrary field extension K/k, any projective, normal k-variety having function field K can be obtained using a standard atlas defined by a transcendence basis of K/k.

Ideal Sheaves and Blow-Ups

We define an **ideal sheaf $\mathcal{I}$ on $\mathfrak{ZR}(K|k)$** by giving an ideal $\mathcal{I}(\mathfrak{U})$ of $\mathcal{O}(\mathfrak{U})$, for each chart $\mathfrak{U}$. Under mild conditions, $\mathcal{I}$ induces a sheaf of ideals $\mathcal{I}_{|X}$ on any $X \in \mathrm{ob}\,(\underline{\mathtt{PROPNORM}}(K|k))$. Moreover $\mathcal{I}$ is compatible with pullbacks in the sense that if $f \colon Y \to X$ is a birational morphism, then $\mathcal{I}_{|Y} = f^{-1}\mathcal{I}_{|X}$.
Conversely, given an ideal sheaf $\mathfrak{I}$ on a proper normal k-variety X with function field K, we can define a sheaf of ideals $\tilde{\mathfrak{I}}$ on $\mathfrak{ZR}(K|k)$ such that $\tilde{\mathfrak{I}}_{|X} = \mathfrak{I}$.

Example: Given a finite tuple $\mathbf{f}$ of elements of K, we define the sheaf of ideals $\mathcal{Id}_{\mathbf{f}}$ on $\mathfrak{ZR}(K|k)$ by setting $\mathcal{Id}_{\mathbf{f}}(\mathfrak{U}) = \overline{\mathbf{f}\mathcal{O}(\mathfrak{U}) \cap \mathcal{O}(\mathfrak{U})}$, for any chart $\mathfrak{U}$.

It turns out that any sheaf of ideals on a projective, normal variety X can be realized as $\mathcal{Id}_{\mathbf{f}|X}$, for some finite tuple $\mathbf{f}$ in K.

Using this description of ideal sheaves, we are able to explicitly find the atlas which defines the blow-up of a projective, normal variety X at a proper closed subscheme.

Quasi-coherent sheaves of modules

We define a **sheaf of modules $\mathcal{M}$ on $\mathfrak{ZR}(K|k)$** by giving a $\mathcal{O}(\mathfrak{U})$-module $\mathcal{M}(\mathfrak{U})$, for each chart $\mathfrak{U}$. As in the case of ideals, $\mathcal{M}$ induces a quasi-coherent sheaf of modules $\mathcal{M}_{|X}$ on any $X \in \mathrm{ob}\,(\underline{\mathtt{PROPNORM}}(K|k))$ which is compatible with pullbacks and any $\mathcal{O}_X$-module $\mathfrak{M}$ gives rise to a sheaf of modules $\widetilde{\mathfrak{M}}$ on $\mathfrak{ZR}(K|k)$ such that $\widetilde{\mathfrak{M}}_{|X} = \mathfrak{M}$.

Additionally $\mathcal{M}_{|X}$ is compatible with all the tensor operations on sheaves.

Questions:

- What is the sheaf on $\mathfrak{ZR}(K|k)$ inducing the canonical module (at least on smooth varieties)?
- Can we determine classical birational invariants of a variety (like the geometric genus or the plurigenera) by examining the atlas that defines it?