The Generators, Relations and Type of the Backelin Semigroup

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Background

Let $H = \langle a_1, a_2, \ldots, a_h \rangle$ be a *numerical semigroup*. The integer h is called the *embedding dimension* of H, and we study the case where $h = 4$.
To this H, one naturally associates a *numerical semigroup ring* $K[H] = K[t^a : a \in H]$.
There is a natural ring homomorphism $\phi : K[x_1, \ldots, x_h] \to K[H]$, given by $\phi(x_i) = t^{a_i}$.
Consider a $\mathbb{N}$-grading on $K[a_1, \ldots a_h]$ such that the degree of x_i is a_i.
$\ker(\phi)$ is called the *presentation ideal* of H denoted by I_H. It is clear that $K[x_1, \ldots, x_h]/I_H \cong K[H]$.
In fact,

$$I_H = (x^u - x^v \ : \ u, v \in \mathbb{N}^n, \sum_{i=1}^{h} u_i a_i = \sum_{i=1}^{h} v_i a_i)$$

where for $u = (u_i)_{i=1}^h$, we let $x^u = x_1^{u_1} \ldots x_h^{u_h}$. The minimal number of generators of I_H, $\mu(I_H)$ is the first Betti number of $K[H]$ or $\beta_1(K[H])$, while $\beta_0(K[H]) = 1$. In our case, $projdim(K[H]) = 3$, so there are two more Betti numbers β_2 and β_3 that can be naturally associated to H. The third Betti number is known as the *type* of $K[H]$, and is given by the cardinality of the set $\{x \in \mathbb{Z} \setminus H : x + h \in H \ \forall h \in H\}$.
The *tangent cone* of H is given by the associated graded ring of $K[H]$ with respect to the homogenous maximal ideal. Its Betti numbers form an upper bound to the Betti numbers of $K[H]$. When equality holds, $K[H]$ is said to be of *homogeneous type*.

Historical Context

When $h = 2$, I_H is principal.
When $h = 3$, Herzog showed that $\mu(I_H) \leq 3$, and in fact I_H can be described precisely
When $h = 4$, Bresinsky showed that that there is no upper bound on $\mu(I_H)$. Specifically, Bresinsky's semigroup

$$H = \langle (2n-1)2n, (2n-1)(2n+1), 2n(2n+1), 2n(2n+1) + 2n - 1 \rangle$$

for $n \geq 2$ has $\mu(I_H) = 4n$. There is another example in the literature provided by Arslan for which $\mu(I_H) = 2n+2$. A natural question is: *Are there such examples where $\mu(I_H)$ is odd while arbitrarily large?*

In this project, we study Backelin's family of semigroups defined as follows $n \geq 2$, $r \geq 3n+2$, let

$$H = H_{n,r} = \langle r(3n+2) + 3, r(3n+2) + 6, r(3n+2) + 3n + 4, r(3n+2) + 3n + 5 \rangle$$

Fröberg, Gottlieb, and Häggkvist have communicated this example as the first family with fixed embedding dimension, but with unbounded type, attributing the claim to Backelin. They verify the claim by showing that the type is at least $2n+2$, and also state, however incorrectly, that the type equals $2n+3$

Relevance and Goals of the Paper

We produce an explicit minimal generating set for I_H and prove therefore that $\mu(I_H) = 3n+4$, which was stated by Stamate, based on numerical evidence obtained with Singular and GAP. This particular family provides a first example where the number $\mu(I_H)$ is odd for infinitely many values of n, in this case for all n odd.
Secondly, we verify that the type of this semigroup is $3n+2$ correcting a wrong claim in the literature. This discrepancy was noted by Stamate in his paper, based upon computer algebra computations, but we verify it here in full generality.
Moreover, there has been interest in the literature in numerical semigroup rings for which the Betti numbers coincide to the Betti numbers of their tangent cone. We show that this is the case for the Backelin family of numerical semigroups, confirming numerical evidence.

Minimal Generating set

Consider the following sets of elements in $K[x, y, z, w]$:

$$S_1 = \{x^{n-k} z^{3k-1} - y^{n-k+1} w^{3k-2} : k = 1, \ldots, n\},$$

$$S_2 = \{x^{r-k+2} y^k - z^{3(n-k)-1} w^{r-3(n-k)+2} : k = 1, \ldots, n-1\},$$

$$S_3 = \{x^{r-(n+k)+3} y^{n+k} - z^{3(n-k)+1} w^{r-3(n-k)+1} : k = 1, \ldots, n\},$$

$$E = \{xw^3 - yz^3, x^n w^2 - y^{n+1} z, x^{r+2} - yw^r, x^{r-n+2} y^n z - w^{r+2}, x^{2n-1} zw - y^{2n+1}\}.$$

Results and Techniques

The two central theorems of the paper are as follows:

Theorem 1 *The set $\Lambda = S_1 \cup S_2 \cup S_3 \cup E$ generates the presentation ideal of $K[H]$.*

Theorem 2 *(1) A minimal generating set of the presentation ideal of the semigroup ring $K[H]$ is given by $S_1 \cup S_2 \cup S_3 \cup E$.*
(2) The type of $K[H]$ is $3n+2$ and the sequence of Betti numbers of $K[H]$ is $(1, 3n+4, 6n+5, 3n+2)$.

Assuming Theorem 1, Proving Theorem 2 (1) makes use of a Theorem by Herzog that allows one to pass into $K[x, y, z]$ (by sending w to 0 under the canonical map π) and, ensures that the minimal generation of $\pi(I)$ by images of $f_i \in I$ (say g_i) implies the minimal generation of I by f_i, as long as $\deg(f_i^*) = \deg(g_i^*)$, where f^* and g^* denote the inital forms of f_i and g_i.

Proving Theorem 2 (2) boils down to computing the type of $\pi(I_H) = \bar{I}$ which is done by computing the set of monomials whose images form a basis for $\frac{\bar{I}:(x,y,z)}{\bar{I}}$ and this gives us a set of $3n+2$ monomials, thereby ensuring that the type of $K[H]$ is $3n+2$.

Proving Theorem 1 makes up a bulk of the paper and is done in a case by case basis, with the underlying inductive argument that given $f \in I_H$, either $f \in \Lambda$ or the homogeneous degree of f can be reduced using a polynomial in (Λ). Provided below is an example of such a reduction.

Example 1 *Take $xyw^{6n-1} - z^{6n+1} \in I_H$. Now, Notice that we have $z^{3n-1} - yw^{3n-2}$ and $xw^3 - yz^3 \in \Lambda$. So, we write $(xyw^{6n-1} - z^{6n+1}) - z^{3n+2}(z^{3n-1} - yw^{3n-2}) = yw^{3n-2}(xw^{3n+1} - z^{3n+2})$. Now consider $xw^{3n+1} - z^{3n+2} \in I_H$, and we can write $xw^{3n+1} - z^{3n+2} + z^3(z^{3n-1} - yw^{3n-2}) = xw^3 - yz^3 \in \Lambda$. Notice that at every step of the equality, the homogeneous degree of the binomial dropped. Working backwards, we conclude that $xyw^{6n-1} - z^{6n+1} \in (\Lambda)$.*

Further Directions

While this project particularly dealt with the Backelin semigroup, there is a need for a better theoretical framework that helps produce other examples. Other directions of enquiry are finding a minimal generating set that also serves as a Gröbner basis under some order, and producing a minimal free resolution of the semigroup ring. This has been done for a few other famous examples in the literature.

References

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