Resolution and Tor Algebra Structures for Certain Ideals Defining Compressed Rings

Keller VandeBogert

University of South Carolina

Definitions/Setup

- Let $n \geq 1$ be an integer and k denote a field of arbitrary characteristic.
- Let V be a vector space of dimension n over k.
- Give the symmetric algebra $S(V)$ and divided power algebra $D(V^*)$ the standard grading (that is, $S_1(V) = V$, $D_1(V^*) = V^*$).
- If I is a homogeneous ideal with associated inverse system minimally generated by elements $\phi_1, \ldots, \phi_k$ with $\deg \phi_i = s_i$, then there are induced vector space homomorphisms
$$\Phi_i : S_i \to \bigoplus_{j=1}^{k} D_{s_j - i}$$
sending $f \mapsto (f \cdot \phi_1, \ldots, f \cdot \phi_k)$.

Definition. The ideal I defines a *compressed* ring if the maps Φ_i have maximal rank for all i.
It can be shown that if $I \subseteq k[x,y,z]$ defines a grade 3 compressed ring with socle $k(-s_1)^\ell \oplus k(-s_2)$ for some $\ell \geq 1$, then $s_2 < 2s_1$.

Questions

1. What is the structure of grade 3 homogeneous ideals defining compressed rings occupying the "boundary," that is, with $\operatorname{Soc}(R/I) = k(-s)^\ell \oplus k(-2s+1)$?
2. Can we find a resolution for all such ideals?
3. Do these ideals satisfy any other interesting properties?

Grade 3 Gorenstein Ideals of Odd Socle Degree

It turns out that the behavior of ideals as in Question 1 is closely related to that of grade 3 Gorenstein ideals with odd socle degree.

Theorem ([1])

Let $K \subseteq k[x,y,z]$ be a homogeneous grade 3 Gorenstein ideal with R/K compressed and $\operatorname{Soc}(R/K) = k(-2s+1)$ for some integer s. Then R/K has Betti table

	0	1	2	3
0	1	·	·	·
$s-1$	·	$s+1$	b	·
s	·	b	$s+1$	·
$2s-1$	·	·	·	1

where b is some integer. Moreover, $b \leq s$.
If s is even and I is chosen generically, then $b = 0$.
If s is odd and I is chosen generically, then $b = 1$.

Idea of Proof: The values $s+1$ and b follow from work of Boij. The fact that $b \leq s$ uses Boij-Söderberg theory, and it can also be shown that generically, b attains the minimal possible value. One then enumerates explicit ideals attaining the desired values for b and argues that no smaller values can be attained.

Proposition ([2])

Let $I \subseteq R = k[x,y,z]$ be a grade 3 homogeneous ideal defining a compressed ring with $\operatorname{Soc}(R/I) = k(-s)^\ell \oplus k(-2s+1)$. Then there exists a grade 3 Gorenstein ideal $I_t = (\phi_1, \ldots, \phi_{s+1}, \psi_1, \ldots, \psi_b)$ defining a compressed ring with $\operatorname{Soc}(R/I_t) = k(-2s+1)$ such that
$$I = (\phi_1, \ldots, \phi_{s+1-\ell}, \psi_1, \ldots, \psi_b) + (x,y,z) \cdot (\phi_{s+2-\ell}, \ldots, \phi_{s+1})$$

Idea of Proof: Using the theory of inverse systems, I may be written as the intersection $I_t \cap I'$, where I_t is a grade 3 Gorenstein ideal defining a compressed ring with $\operatorname{Soc}(R/I_t) = k(-2s+1)$. The compressed hypothesis allows us to deduce the generating set of I in terms of the generating set of I_t.

The Resolution

- The minimal free resolution of grade 3 Gorenstein ideals is well known, due to a result of Buchsbaum and Eisenbud.
- The minimal free resolution of the ideal (x,y,z) is the Koszul complex.

Given the data of an ideal $K = (\phi_1, \ldots, \phi_n)$ and another ideal J, a resolution of the quotient defined by $(\phi_1, \ldots, \phi_{n-1}) + J\phi_n$ can be built using the machinery of trimming complexes as introduced in ([3]). Combining this with the previous Proposition yields:

Theorem ([2])

Let I be as before. Then R/I is resolved by an iterated trimming complex as in ([3, Theorem 3.4]). If s is even and I is chosen generically, then this resolution is minimal.

- Even if the above resolution is not minimal, one can deduce the graded Betti numbers of R/I, and hence bound certain parameters arising in the Tor algebra classification.

Tor Algebra Structure

If I is an ideal, then the module
$$\operatorname{Tor}_\bullet^R(R/I,k) := \bigoplus_i \operatorname{Tor}_i^R(R/I,k)$$
admits the structure of an associative k-algebra.

- A complete classification of the Tor algebra structures for quotient rings of projective dimension 3 was given by Weyman and Avramov, Kustin, and Miller.
- One of the classes arising in this classification is the class $G(r)$, where r is a parameter telling us about the largest subalgebra exhibiting Poincaré duality.

Theorem ([2])

Let $I \subseteq k[x,y,z]$ be a grade 3 homogeneous ideal defining a compressed ring with $\operatorname{Soc}(R/I) = k(-s)^\ell \oplus k(-2s+1)$, and let b be as in the first Theorem. If $\ell \leq s+b-1-\min\{\ell b, 3\}$, then I has Tor algebra class $G(\mu(I) - 3\ell)$.

- In 2012, Avramov had conjectured that if R/I defines a ring of Tor algebra class $G(r)$, then I is Gorenstein.
- The first counterexamples to this conjecture were produced by Christensen, Veliche, and Weyman. Their counterexamples were all of type 2.
- Observe that the above Theorem implies that for any $r \geq 2$, there exist rings of *arbitrarily* large type with Tor algebra class $G(r)$.

References

[1] Keller VandeBogert.
Structure theory for a class of grade 3 homogeneous ideals defining type 2 compressed rings.
To Appear, J. Comm. Alg., arXiv:1912.06949, 2019.

[2] Keller VandeBogert.
Resolution and tor algebra structures of grade 3 ideals defining compressed rings.
arXiv:2004.06691, 2020.

[3] Keller VandeBogert.
Trimming complexes and applications to resolutions of determinantal facet ideals.
arXiv:2004.06016, 2020.

Contact Information

people.math.sc.edu/kellerlv
kellerlv@math.sc.edu